Baltic Way

12. November 2022

Version: *English*

Time allowed: 4 hours and 30 minutes.
During the first 30 minutes, questions may be asked.
Tools for writing and drawing are the only ones allowed.

Problem 1. Let $\mathbb{R}^+$ denote the set of positive real numbers. Assume that $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a function satisfying the equations

$$f(x^3) = f(x)^3 \text{ and } f(2x) = f(x)$$

for all $x \in \mathbb{R}^+$. Find all possible values of $f\left(\sqrt[2022]{2}\right)$.

Problem 2. We define a sequence of natural numbers by the initial values $a_0 = a_1 = a_2 = 1$ and the recursion

$$a_n = \left\lfloor \frac{n}{a_{n-1}a_{n-2}a_{n-3}} \right\rfloor$$

for all $n \geq 3$. Find the value of a_{2022}.

Problem 3. We call a two-variable polynomial $P(x, y)$ *secretly one-variable*, if there exist polynomials $Q(x)$ and $R(x, y)$ such that $\deg(Q) \geq 2$ and $P(x, y) = Q(R(x, y))$ (e.g. $x^2 + 1$ and $x^2y^2 + 1$ are secretly one-variable, but $xy + 1$ is not).

Prove or disprove the following statement: If $P(x, y)$ is a polynomial such that both $P(x, y)$ and $P(x, y) + 1$ can be written as the product of two non-constant polynomials, then P is secretly one-variable.

Note: All polynomials are assumed to have real coefficients.

Problem 4. The positive real numbers x, y and z satisfy $xy + yz + zx = 1$. Prove that

$$2(x^2 + y^2 + z^2) + \frac{4}{3}\left(\frac{1}{x^2+1} + \frac{1}{y^2+1} + \frac{1}{z^2+1}\right) \geq 5.$$

Problem 5. Let $\mathbb{R}$ denote the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0)+1 = f(1)$, and for any real numbers x and y,

$$f(xy - x) + f(x + f(y)) = yf(x) + 3.$$

Problem 6. Mattis is hosting a badminton tournament for 40 players on 20 courts numbered from 1 to 20. The players are distributed with 2 players on each court. In each round a winner is determined on each court. Afterwards, the player who lost on court 1, and the player who won on court 20 stay in place. For the remaining 38 players, the winner on court i moves to court $i + 1$ and the loser moves to court $i - 1$. The tournament continues until every player has played every other player at least once. What is the minimal number of rounds the tournament can last?

Problem 7. The writer Arthur has $n \geq 1$ co-authors who write books with him. Each book has a list of authors including Arthur himself. No two books have the same set of authors. At a party with all his co-author, each co-author writes on a note how many books they remember having written with Arthur. Inspecting the numbers on the notes, they discover that the numbers written down are the first n Fibonacci numbers (defined by $F_1 = F_2 = 1$ and $F_{k+2} = F_{k+1} + F_k$). For which n is it possible that none of the co-authors had a lapse of memory?

Problem 8. For a natural number $n \geq 3$, we draw $n-3$ internal diagonals in a non self-intersecting, but not necessarily convex, n-gon, cutting the n-gon into $n-2$ triangles. It is known that the value (in degrees) of any angle in any of these triangles is a natural number and no two of these angle values are equal. What is the largest possible value of n?

Problem 9. Five elders are sitting around a large bonfire. They know that Oluf will put a hat of one of four colours (red, green, blue or yellow) on each elder's head, and after a short time for silent reflection each elder will have to write down one of the four colours on a piece of paper. Each elder will only be able to see the colour of their two neighbours' hats, not that of their own nor that of the remaining two elders' hats, and they also cannot communicate after Oluf starts putting the hats on.

Show that the elders can devise a strategy ahead of time so that at most two elders will end up writing down the colour of their own hat.

Problem 10. A natural number a is said to be *contained* in the natural number b if it is possible to obtain a by erasing some digits from b (in their decimal representations). For example, 123 is contained in 901523, but not contained in 3412.

Does there exist an infinite set of natural numbers such that no number in the set is contained in any other number from the set?

Problem 11. Let ABC be a triangle with circumcircle Γ and circumcentre O. The circle with centre on the line AB and passing through the points A and O intersects Γ again in D. Similarly, the circle with centre on the line AC and passing through the points A and O intersects Γ again in E. Prove that BD is parallel with CE.

Problem 12. An acute-angled triangle ABC has altitudes AD, BE and CF (where D, E and F lie on the sides BC, CA and AB, respectively). Let Q be an interior point of the segment AD, and let the circumcircles of the triangles QDF and QDE meet the line BC again at points X and Y, respectively. Prove that $BX = CY$.

Problem 13. Let $ABCD$ be a cyclic quadrilateral with $AB < BC$ and $AD < DC$. Let E and F be points on the sides BC and CD, respectively, such that $AB = BE$ and $AD = DF$. Let further M denote the midpoint of the segment EF. Prove that $\angle BMD = 90°$.

Problem 14. Let Γ denote the circumcircle and O the circumcentre of the acute-angled triangle ABC, and let M be the midpoint of the segment BC.

Let T be the second intersection point of Γ and the line AM, and D the second intersection point of Γ and the altitude from A. Let further X be the intersection point of the lines DT and BC. Let P be the circumcentre of the triangle XDM. Prove that the circumcircle of the triangle OPD passes through the midpoint of XD.

Problem 15. Let Ω be a circle, and $B \neq C$ two fixed points on Ω. Given a third point A on Ω, let X and Y denote the feet of the altitudes from B and C, respectively, in the triangle ABC. Prove that there exists a circle Γ such that XY is tangent to Γ regardless of the choice of the point A.

Problem 16. Let $\mathbb{Z}^+$ denote the set of positive integers. Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ satisfying the condition

$$f(a) + f(b) \mid (a+b)^2$$

for all $a, b \in \mathbb{Z}^+$.

Problem 17. Let n be a positive integer such that the sum of its positive divisors is at least $2022n$. Prove that n has at least 2022 distinct prime factors.

Problem 18. Find all pairs (a, b) of positive integers such that $a \leq b$ and

$$\gcd(x, a)\gcd(x, b) = \gcd(x, 20)\gcd(x, 22)$$

holds for every positive integer x.

Problem 19. Find all triples (x, y, z) of nonnegative integers such that

$$x^5 + x^4 + 1 = 3^y 7^z.$$

Problem 20. Ingrid and Erik are playing a game. For a given odd prime p, the numbers $1, 2, 3, \ldots, p-1$ are written on a blackboard. The players take turns making moves with Ingrid starting. A move consists of one of the players crossing out a number on the board that has not yet been crossed out. If the product of all currently crossed out numbers is $1 \pmod{p}$ after the move, the player whose move it was receives one point, otherwise zero points are awarded. The game ends after all numbers have been crossed out.

The player who has received the most points by the end of the game wins. If both players have the same score, the game ends in a draw. For each p, determine which player (if any) has a winning strategy.