LATVIJAS UNIVERSITĀTE

LATVIJAS UNIVERSITĀTE
Eksakto zinātņu un tehnoloģiju fakultāte
A. Liepas Neklātienes matemātikas skola

Open Mathematical Olympiad

Grade 5

5.1. Fill each cell of the grid (see Figure 1) with a digit from 1 to 5. Each row and column must contain every number from 1 to 5 exactly once. Each shape, outlined with a bold line and containing two or three cells, is associated with a number and an operation. Using the numbers written in the cells of that shape and performing the given operation, the result must equal the given number. It is sufficient to demonstrate one example of how this can be achieved. For three of the cells with a bold outline, a number is already written.

For example, in the shape in the upper-left corner marked "2−", the difference between the two numbers in those cells (subtracting the smaller number from the larger) must equal 2.

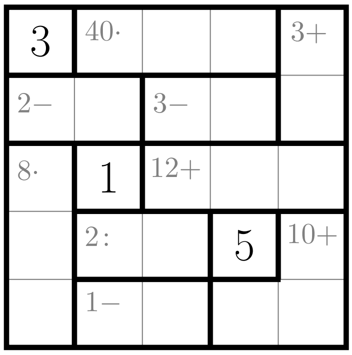

Figure 1

5.2. A box contains 7 green, 3 red, 11 black, and 9 white balls. What is the smallest amount of balls Alexander must take out of the box to guarantee that: **a)** Alex has at least two balls of the same colour; **b)** he has at least two black balls; **c)** he has at least one ball of each colour?

5.3. Write 10 expressions, each using the numbers 3, 5, 7, and 9 exactly once, with exactly three arithmetic operations (addition, subtraction, multiplication, or division) so that the values of these expressions are the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Parentheses may be used for grouping operations, and the order of the numbers may be changed. Division must result in an exact value without rounding. It is sufficient to find one valid expression for each of the numbers. For example, $7 \cdot (9-5) - 3 = 25$ is a valid expression.

5.4. A rectangle is made up of 7×9 squares. Can nine of the shapes shown in Figure 2 be placed within the rectangle? The shapes may be rotated and flipped, but they cannot overlap or extend outside the rectangle.

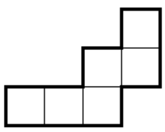

Figure 2

5.5. Andris forgot his watch at home. At a bus stop he noticed 4 people standing there. He asked each of them: "Excuse me, what time is it?" The answers were as follows:

A: "Six minutes to three."

B: "Three minutes to three"

C: "Three minutes past three."

D: "Two minutes past three."

It turned out that none of these people's watches showed the correct time. Moreover, one watch was off by 2 minutes, another by 3 minutes, another by 4 minutes, and another by 5 minutes (it is not known which watch was running late and which was fast). What was the actual time? Justify your answer!

Open Mathematical Olympiad

Grade 6

6.1. Fill each cell of the grid (see Figure 1) with a digit from 1 to 6. Each row and column must contain every number from 1 to 6 exactly once. Each shape, outlined with a bold line and containing two or three cells, is associated with a number and an operation. Using the numbers written in the cells of that shape and performing the given operation, the result must equal the given number. It is sufficient to demonstrate one example of how this can be achieved. For one of the cells with a bold outline, a number is already written.

For example, in the shape in the upper-left corner marked "4−,", the difference between the two numbers in those cells (subtracting the smaller number from the larger) must equal 4.

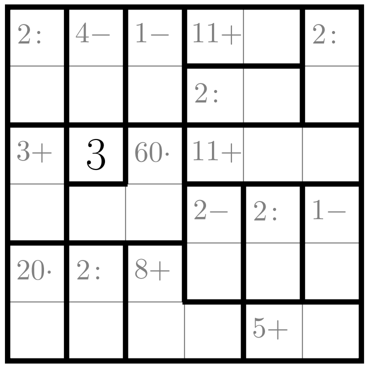

Figure 1

6.2. A box contains 14 violet, 18 brown, 4 orange and 19 yellow balls. Anna drew 9 balls from the box.

- **a)** Is it certain that exactly five of the balls drawn are yellow?
- **b)** Is it certain that Anna drew at least three balls of the same colour?
- **c)** What is the smallest amount of additional balls Anna must draw from the box to ensure that she has at least six balls of the same colour?

6.3. Each of the 10 dwarves either always tells the truth or always lies. It is known that each dwarf has one favourite ice cream flavour: vanilla, chocolate, or strawberry.
Snow White asked the dwarves to raise their hands if their favourite ice cream flavour was vanilla, and eight of them raised their hands. Then she asked about chocolate ice cream, and half of them raised their hands. When she asked about strawberry ice cream, only one dwarf raised their hand. How many of these dwarves always tell the truth?

6.4. Show how, by cutting along the grid lines, the shape given in Figure 2 can be divided into two equal parts. The parts are considered equal if, when rotated or flipped, one can be placed on top of the other so that they match.

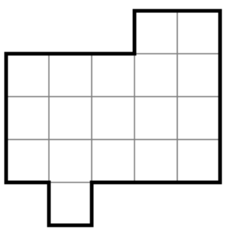

Figure 2

6.5. In a transparent box there are sweets, and four children must guess the number of sweets in it. Each one offers their guess: 116, 124, 120, and 128. It turns out that none of them guessed correctly.
Determine how many sweets could be in the box, given that it is known that, for the actual number of sweets, one of the children was off by one, one by three, one by five, and one by nine.

Open Mathematical Olympiad

Grade 7

7.1. Marta, Sandris and Linda want to prepare the clubroom for a Christmas party. It is known that Marta could do it by herself in one hour, Sandris could do it in one and a half hours, and Linda could do it in three hours. Marta arrived at the clubroom at 16:00, Sandris arrived 10 minutes later, and Linda arrived 15 minutes after Sandris (each of them started preparing the room immediately upon arrival). At what time was the clubroom ready?

7.2. Jurģis distributed 11 prizes at the Michaelmas market through a lottery. Each prize contains 6 autumn delights: apples, pears, and beets. It is known that each prize contains at least one apple, one pear, and one beet. Prove that at least two of the prizes must have had the same contents.

7.3. The first term of a number sequence is 12. Each subsequent term is obtained by either multiplying the previous term by 2 or 3, or dividing it by 2 or 3 (if it divides evenly). Can the 61st term of this number sequence be 54?

7.4. A grid square is made up of 10×10 smaller squares. What is the largest amount of shapes shown in Figure 1 that can be cut from this square, if the cutting lines lines must follow grid lines. The shapes can be rotated.

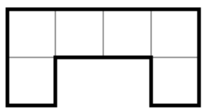

Figure 1

7.5. Anita, Maija, Ināra, and Sandra performed at a concert. Each song was sung by three girls. How many songs did the girls sing in total, given that Anita sang 7 songs (more than any other girl), and Sandra sang 4 songs (fewer than any other girl)?

Open Mathematical Olympiad

Grade 8

8.1. Given three distinct real numbers, it is known that the arithmetic mean of the arithmetic mean of the two smallest numbers and the arithmetic mean of the two largest numbers is equal to the arithmetic mean of all three numbers. Furthermore, the arithmetic mean of the largest and the smallest number is 2024. Determine the sum of these three numbers.

8.2. The integers from 1 to 10 are written in an arbitrary order around a circle. Prove that there must be three consecutive numbers whose sum is at least 17.

8.3. Three wizards can transform numbers using a ritual, but each wizard knows only one spell:

- the first wizard can subtract 1 from any number;
- the second wizard can divide any number by 2;
- the third wizard can multiply any number by 3.

To transform a number, the wizards can apply their spells in any order, even skipping other wizards. However, each wizard can use their spell only 5 times in each ritual, and the intermediate result must be an integer that does not exceed 9. Can the wizards, within a single ritual, transform the numbers $3, 8, 9, 2, 4$ into: **a)** $3, 3, 3, 3, 3$; **b)** $5, 5, 5, 5, 5$?

8.4. Points A, B and C are placed on a circle with centre O, such that the point O lies within the triangle ABC. It is known that $\sphericalangle AOC = \alpha$ and $\sphericalangle OAB = \beta$. Express the angle $\sphericalangle BCO$ in terms of α and β.

8.5. Five heavy boxes are given, and they are arranged as shown in Figure 1. These boxes can only be moved by rotating them 90 degrees around one of the corners. The boxes cannot be moved on top of other boxes. After several such movements, the boxes were arranged as shown in Figure 2. Which of these boxes could have initially been in the centre of the original arrangement in Figure 1? An example of how a box can be moved around a corner in two different ways can be seen in Figure 3

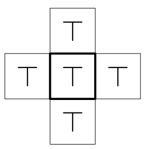

Figure 1

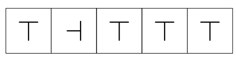

Figure 2

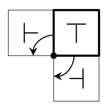

Figure 3

Open Mathematical Olympiad

Grade 9

9.1. Let a and b be real numbers such that

$$\frac{a}{a^2-5} = \frac{b}{5-b^2} = \frac{ab}{a^2b^2-5}.$$

What is the value of the expression a^4+b^4, if we additionally know that $a+b \neq 0$?

9.2. Each of the 28 students in the class received a grade on a test, which is an integer between 0 and 10. Prove that either at least 4 students have the same grade, or at least 4 students received a grade higher than 7.

9.3. Three numbers, $11, 12, 13$ are written on a blackboard. In one move, Agnese can choose one of the numbers, erase it, and write the number obtained by doubling the sum of the other two numbers and then subtracting the chosen number. Can Agnese, by repeating such moves, achieve that the numbers $20, 24, 25$ are written on the blackboard?

9.4. On the parallelogram $ABCD$, points E and F are marked on the sides BC and CD, respectively. The intersection of segments AE and BF is G, the intersection of segments AF and ED is I, and the intersection of segments BF and ED is H. Prove that $S_{AGHI} = S_{BEG} + S_{CEHF} + S_{DFI}$.

9.5. Inga has a phone with the following button layout:

1	2	3
4	5	6
7	8	9
	0	

Her friend Zane's 9-digit phone number has the following properties:

- all digits in Zane's phone number are distinct;
- the first four digits are arranged in increasing order, and the centres of the corresponding buttons form a square;
- the centres of the buttons for the last four digits also form a square;
- Zane's phone number is divisible by 15.

How many nine-digit phone numbers could be Zane's phone number?

Open Mathematical Olympiad

Grade 10

10.1. Find all natural numbers m and n such that $m^3n + m + n = mn + 2mn^2$.

10.2. Given 15 three-digit numbers, prove that among these numbers, there must be two such that the sum of their digits is the same, or two such that the sum of the sums of their digits equals 28.

10.3. Anniņa wrote n distinct natural numbers on a blackboard and for each pair of numbers on the board, she calculated their sum. Upon examining these sums, it turned out that each digit from 0 to 9 appears at least once as the last digit of one of the sums. What is the smallest possible value of n?

10.4. A point E is placed inside the square $ABCD$ such that $\sphericalangle CBE = 15°$ and $AE = ED$. Prove that the triangle AED is equilateral.

10.5. Jānītis is lost in a forest with an area of S square kilometres. The only information known about the forest is that it is located on a plane and it does not contain any holes. Prove that there exists a strategy with a route that is no longer than $2\sqrt{\pi S}$ kilometres, by which Jānītis can always get out of the forest (Jānītis does not know the shape of the forest or his initial location within it).

Open Mathematical Olympiad
Grade 11

11.1. Given a triangle with side lengths a, b and c, prove that the following inequality holds:

$$\frac{a^2+2bc}{b^2+c^2}+\frac{b^2+2ac}{a^2+c^2}+\frac{c^2+2ab}{a^2+b^2}>3.$$

11.2. There were 95 participants at a chess festival. It is known that during the festival, each participant played no more than 10 games, and everyone lost at least one game. In each game, the winner received one point, the loser received 0 points, and if the game ended in a draw, both players received half a point. Can we be sure that at the end of the festival there were at least **a)** 5; **b)** 6 participants with the same amount of points?

11.3. A four-digit number written on a blackboard, with no digit being zero. It is known that if any digit of the number is erased, the remaining three-digit number is not divisible by 3. Prove that it is possible to erase two of its digits so that the remaining two-digit number is divisible by 3.

11.4. The diagonals of a convex quadrilateral divide it into four triangles with equal perimeters. Prove that this quadrilateral is a rhombus.

11.5. Several chameleons are sitting around a circular table. Each chameleon can change its colour to red or green. Every minute, a chameleon changes its colour if its two neighbours are different colours. Is it guaranteed, regardless of the initial colouring of the chameleons, that at some point every chameleon will return to its starting colour if there are **a)** 6; **b)** 7 chameleons sitting at the table?

Open Mathematical Olympiad

Grade 12

12.1. Given a triangle with side lengths a, b and c, prove that the following inequality holds:

$$a + b + c > \sqrt{2(a^2 + b^2 + c^2)}.$$

12.2. There were 64 participants at a chess festival. It is known that during the festival, each participant played exactly 12 games, no one won all 12 games, but everyone won at least one game. In each game, the winner received one point, the loser received 0 points, and if the game ended in a draw, both players received half a point. Can we be sure that at the end of the festival there were at least **a)** 3; **b)** 4 participants with the same amount of points?

12.3. Several integers are written on a blackboard, and the sum of their cubes is 2024. Is it possible that the sum of these numbers is **a)** 24; **b)** 26?

12.4. A quadrilateral $ABCD$ is inscribed in a circle such that $\sphericalangle BAD = 2\sphericalangle ADC$ and $CD = 2BC$. A point H is placed on the side AD such that $\sphericalangle DHC = 90°$. Prove that $BH \parallel CD$.

12.5. Jānītis is lost in a forest in the shape of a convex polygon with an area of S square kilometres. Prove that there exists a strategy with a route that is no longer than $\sqrt{2\pi S}$ kilometres, by which Jānītis can always get out of the forest (Jānītis does not know the shape of the forest or his initial location within it).