The LAIMA series

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DIRICHLET PRINCIPLE

Part I

Theory, Examples, Problems

Experimental Training Material

Translated from Latvian by M. Kvalberga

Rīga 2005

A. Andžāns, B. Johannesson. Dirichlet Principle Part I.
Rīga: Mācību grāmata, 2005. - 81 pp.

The book is an advanced teaching aid in mathematics for elementary school students. It contains theoretical considerations, examples and problems for independent work. It can be used as a supplementary text in classroom or for individual studies, including preparation to mathematical olympiads.

The final version was prepared by Ms. Inese Bērziņa

ISBN 9984-18-069-7

Reģ. apl. No. 2-0266

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About LAIMA series

In 1990 international team competition “Baltic Way” was organized for the first time. The competition gained its name from the mass action in August, 1989, when over a million of people stood hand by hand along the road Tallin - Riga - Vilnius, demonstrating their will for freedom.

Today “Baltic Way” has all the countries around the Baltic Sea (and also Iceland) as its participants. Inviting Iceland is a special case remembering that it was the first country all over the world, which officially recognized the independence of Lithuania, Latvia and Estonia in 1991.

The “Baltic Way” competition has given rise also to other mathematical activities. One of them is project LAIMA (Latvian - Icelandic Mathematics project). Its aim is to publish a series of books covering all essential topics in the area of mathematical competitions.

Mathematical olympiads today have become an important and essential part of education system. In some sense they provide high standards for teaching mathematics on advanced level. Many outstanding scientists are involved in problem composing for competitions. Therefore “olympiad curricula”, considered all over the world, is a good reflection of important mathematical ideas at elementary level.

At our opinion there are relatively few basic ideas and relatively few important topics which cover almost all what international mathematical community has recognized as worth to be included regularly in the search and promoting of young talents. This (clearly subjective) opinion is reflected in the list of teaching aids which are to be prepared within LAIMA project.

Fourteen books have been published so far in Latvian. They are also available electronically at the web - page of Latvian Education Informatization System (LIIS) http://www.liis.lv. As LAIMA is rather a process than a project there is no idea of final date; many of already published teaching aids are second and third versions and will be extended regularly.

Benedikt Johannesson, the President of Icelandic Society of mathematics, inspired LAIMA project in 1996. Being the co-author of many LAIMA publications, he was also the main sponsor of the project for many years.

This book is the first LAIMA publication in English. It was sponsored by the Scandinavian foundation “Nord Plus Neighbours”.

Foreword

This book is intended for the pupils who have an extended interest in mathematics. It can be used also by mathematics teachers and heads of mathematics circles.

For understanding the contents of the book and solving the problems, it is enough to master the course of a 9-year school. The majority of problems and examples can already be solved by 7^{th} form pupils, a great part - even by 5^{th} and 6^{th} formers.

The book contains theoretical material, examples and problems for independent solution. More difficult problems and examples are marked with asterisk **(*)**, entirely difficult - with the letter **"k"**.

We advise the readers to work actively with the book. Having read the example, please try to solve it independently, before you read the solution offered by the authors. However, you must definitely read the authors' solutions - there may turn up ideas and methods of solution, which have been unknown to you before.

The main thing that you should pay attention to, is the general line of the applied judgements, not just abstract formulations of theorems.

In each chapter, among the problems offered for independent solution, there are many problems which differ just in unessential details from the examples analysed in the text. These problems are marked with a small circle **(^{o})**.

After finding a solution or after reading the solution of the example or the problem, always think over:

“Wasn’t it possible to solve the problem otherwise?”

“Could it be possible to prove a stronger (more difficult) result with the same judgement?”

“What similar problems could it be possible to solve by judging the same way?”

We shall give short hints to some problems in the part III.

A wide range of literature has been used in writing this book, also materials of mathematics Olympiads of more than 20 countries. The sources will be mentioned in part III.

As far as we know, it is for the first time that Dirichlet Principle is discussed so widely, sistematizingly and methodically.

Introduction

There are many hundreds of methods developed in mathematics, which are successfully applied in the solution of different problems. The number of such methods is steadily growing. Usually each method is envisaged for solving a comparatively small class of problems, and it is developed in accordance with the peculiarities and specific characters of this class.

However, in mathematics there are also such methods, which are not connected with some specific group of problems; they are used in most different branches. Actually these are not just methods of mathematics, but ways of thinking which people use in solving of mathematical problems as well as in other situations of life. Getting acquainted with such methods is necessary for any intellectual person.

Many theoretical methods in mathematics (and also in life!) are based on such a principle: "**in order to accomplish great things one must concentrate big enough means in at least one direction.**" One must certainly specify the notion "great things", "direction", "big means" in every specific situation. This book shows how to do it in some mathematical situations.

I. INTRODUCTORY PROBLEMS

Let us consider quite a simple problem.

1. example. *Pete has 3 rabbits and 2 hutches. All the rabbits are in hutches. Is there, among these hutches, one with at least 2 rabbits?*

You will certainly answer this question in the affirmative: "Sure! If there wasn't such a hutch Pete would have no more than 1 rabbit in each hutch. So, there would be no more than 2 rabbits in the hutches. As a result, at least 1 rabbit would be at large, which Pete would not tolerate in any case."

But what if Pete had $n+1$ rabbit and n hutches? Is there, among these hutches definitely the hutch with at least 2 rabbits?

You can solve the following problems with similar judgements.

1°. 8 people are sitting round the table. Prove that among them are at least 2 born in the same day of week.

2°. At school there are 370 pupils. Prove that among them you can find 2 pupils born on the same date.

3°. In the class there are 13 pupils. Prove that you can find among them 2 pupils born in the same month.

4°. The only village cinema has the following shows: at 10^{00}, 14^{00}, 16^{00}, 18^{00}, 20^{00}.One day 7 pupils went to the cinema. Prove that at least 2 of the pupils attended the same show.

In all the previous solutions we have used one and the same method based on the theorem.

1-st theorem (Dirichlet Principle).

Theorem D_1.

If more than n objects must be distributed into n groups, then there will definitely be the group containing at least 2 objects.

Let us prove this theorem.

▼ Let us presume the opposite:
no group has more than 1 object. As the total number of groups is n, then no more than n objects are distributed. But we must divide in groups more than n objects. We have got a contradiction, so our presumption is wrong. Then, there is the group which has more than one object, i.e., at least two objects.
The theorem is proved.▲

We can give a more formal exposition for the reader who likes operations with algebraic expressions, equalities, inequalities etc.

▼Let us presume again that there is no more than one object in any group. By denoting the number of objects in i-th group with k_i ($i = 1; 2; \ldots; n$), we get inequalities $k_1 \le 1$; $k_2 \le 1$; $k_3 \le 1$; …; $k_{n-1} \le 1$; $k_n \le 1$. Summing them up, we get $k_1 + k_2 + \ldots + k_n \le 1 + \ldots + 1$ (n times 1) or $k_1 + k_2 + \ldots + k_n \le n$. But $k_1 + k_2 + \ldots + k_n$ is the number of all the objects distributed in groups, so it must exceed n.

We get the contradiction which proves the theorem. ▲

Usually Dirichlet Principle is formulated as follows:

“If more than n rabbits must be put into n hutches, then at least in one hutch there will be more than one (so, at least 2) rabbits.”

Henceforth, we shall refer to this theorem as to $\mathbf{D_1}$.

Applying Dirichlet Principle to problem solving, the very point is to find what will be “the hutch” and what will be “rabbits” in each problem. You can master it only by solving many problems.

In the above mentioned probles the choice could be as follows: in

1^o. problem: “hutches” - days of week, “rabbits” - people round the table;

2^o. problem: “hutches” - different dates (there are 366), “rabbits” - pupils;

3^o. problem: “hutches” - months, “rabbits” - pupils;

4^o. problem: “hutches” - shows, “rabbits” - pupils.

Let us look at still more examples.

2. example. *The antropologysts have stated that a human’s number of hairs can not be bigger than 500000. Prove that in Riga there live at least two persons with the same number of hairs.*

1st solution. Let us try to reason like in the 1st example about rabbits. Only this time in the role of rabbits we have the inhabitants of Riga, but the “hutches” will be formed in the following way (there will be 500001):

In I “hutch” we shall put the bald-headed,
in II “hutch” the persons with exactly 1 hair,
in III “hutch” the persons with exactly 2 hairs,
…
in the 500001st “hutch” the persons with exactly 500000 hairs.

Remember :we must prove that in Riga there live at least 2 persons with the same number of hairs. It means that we must prove: there exist at least one “hutch” where we “have put in” at least two persons.

Let us presume the opposite, that there isn’t such a hutch. So, the 1st “hutch” is empty, or there is only one person there; also the 2nd “hutch” is either empty or there is one person etc.

So, in all “the hutches” there are no more than 1+1+…+1 (500001 times 1) = 500001 person altogether. But in Riga there live more than 800000 persons and every one of them must be in one of the “hutches” formed by us. In result, our presumption has been wrong, and we have proved what the problem required. □

2nd solution. Let us form 500001 group where:

in the 1st group there will be the bald-headed,

in the 2nd group - persons with exactly 1 hair,

in the 3rd group - persons with exactly 2 hairs,

…

in the 500001st group - persons with exactly 500000 hairs.

We must divide all the inhabitants of Riga, which are slightly more than 800000, into these groups. According to $\mathbf{D_1}$ there will definitely be the group with at least 2 persons, what we had to prove. □

Let us take into consideration that it is not declared in the solution of the problem, that in Riga there are **exactly two and no more** inhabitants with the same number of hairs. We just proved that you can find two persons with the same number of hairs amongthe inhabitants of Riga. It does not mean that there cannot be

4, 5 or 100 persons with the same number of hairs in Riga.

We are not interested in the fact **how many** hairs have those two inhabitants we have found, who have the same number of hair; it is only important for us that such persons do exist.

The judgement is based on the fact that in Riga there are **too many** inhabitants to have in each "hutch" no more than one of them.

We must observe that our judgement does not give any indications as to the way of finding the persons with the same number of hair. It just guarantees that there are such persons somewhere in Riga. Let us take into consideration that the solution turned out shorter with the help of Dirichlet Principle.

3. example. *Antennae grow instead of hair on the heads of the inhabitants of Alpha planet: no more than 100000 on each head. According to the last census, there live no less than 8.000.000 inhabitants on this planet. Prove that you will be able to find 80 inhabitants of Alpha planet with the same number of antennae.*

1st solution. In this problem the inhabitants of other planet will be in the role of "rabbits", but we will form "hutches" as follows:

in the I "hutch" we shall put the bald-headed,
in the II "hutch" - those with exactly 1 antenna,
in the III "hutch" - those with exactly 2 antennae,
…
in the 100001st "hutch" - those with exactly 100000 antennae.

Let us presume that we have put all the Alpha inhabitants into “hutches” according to the number of antennae on their heads. Then, paraphrasing the requirement of the problem, we must prove that, definitely, there is the “hutch” with **at least** 80 inhabitants of the planet.

Let us take into consideration that we do not **demand** that there are exactly 80 Alpha inhabitants in the “hutch”. The main thing is that among 100001 “hutch” there is definitely one (no matter which) “hutch” with at least 80 inhabitants of the planet.

Let us presume opposite, that no “hutch” has more than 79 inhabitants of other planet. In such a case we have been lazy and have put into “hutches” no more than $100001 \cdot 79 = 7.900.079$ Alpha inhabitants. 99.921 inhabitants have not been distributed. It is clear that, distributing also these Alpha inhabitants, there will definitely turn up at least one “hutch” with at least 80 inhabitants.□

The solution of this problem could be shorter if we had used the following generalization of Dirichlet Principle.

2-nd theorem (Dirichlet Principle). Theorem D_2.

If more than $m \cdot n$ objects must be divided into groups, then there will definitely be the group with at least m + 1 object.

▼Proof. Let us presume that more than $m \cdot n$ objects are divided into n groups. Let us denote with k_i the number of objects in the i^{th} group, where $i = 1, 2, \ldots, n$, and let us presume the opposite, that in each group there are no more than m objects.

Then in the 1^{st} group there are m or less objects, in the $2^{nd}, 3^{rd}, \ldots, n^{th}$ groups as well

$$\{k_1 \leq m; k_2 \leq m; \ldots; k_m \leq m\}.$$

In result in all the groups together there are only

$$k_1 + k_2 + \ldots + k_n \leq m + m + \ldots + m = m \cdot n$$

objects. But it is stated that more than m · n objects have been divided into groups. So, our presumption has been wrong, and there definitely exists the group with at least m + 1 object. ▲

We shall refer to this theorem as to D_2 and we shall also call it Dirichlet Principle.

2nd solution. (using D_2)

Let us form 100001 group, where

$$8000000 = 79 \cdot 100001 + 99921$$

Alpha inhabitants are distributed depending on the number of antennae (in the I group the bald-headed, in the II group - those with one antenna, …, in the 100001^{st} group - those with 100000 antennae). According to $\mathbf{D_2}$ there exists a group with at least 79 + 1, i.e. at least 80 Alpha inhabitants.□

4. example. *In the house there live 160 inhabitants. None of them is older than 78. Prove that among the inhabitants of the house you can find three ones of the same age.*

1. solution. Let us presume that among the inhabitants of the house you cannot find three ones of the same age. Then, no more than 2 persons' age is 0 years, …, no more than 2 persons are 78. So, in the house there live no more than $2 + 2 + \ldots + 2 = 158$ inhabitants

altogether. But, according to the terms of the problem, there are 160 inhabitants in the house.

So, the presumption is wrong, and in the house there are more than two persons of the same age.□

2. solution. Let us divide all the $79 \cdot 2 + 2 = 160$ inhabitants of the house in to 79 groups, depending on their age. In the 1st group there will be persons whose age is 0, in the 2nd group - those, whose age is 1 year, …, in the 79th group - those, whose age is 78 years.

According to $\mathbf{D_2}$ there will definitely turn up a group with at least 3 persons. Depending on the formation of these groups, it follows that the persons in these groups are of the same age, what we had to prove.□

5. example. *160 inhabitants live in the house; moreover, none of them is older than 78. It is known that at the present moment none of the inhabitants is 0; 1; …; 13; 54; 55; …; 59; 69; 72; 73; …; 76 years old. Prove that you can find four persons of the same age among them.*

Solution. Let us again unite the persons of the same age in one group. Though, this time it is not useful to look at the 79 groups and judge the same way we did in the previous example. This kind of thinking will not lead us to solution. (You can make sure about it yourself.)

Let us take into consideration that no person is included in 26 out of 79 groups. So, let us skip these 26 groups. In result we must prove that among 79 - 26 = 53 groups there is definitely one group including at least 4 persons out of 160 divided into these groups.

If in all groups there would be no more than 3 persons, then in the house there would be no more than $3 \cdot 53 = 159$ persons in total. But, according to the terms of the problem, there are 160 inhabitants in the house. So, our presumption has been wrong.□

All the following problems can be solved by using $\mathbf{D_1}$ or $\mathbf{D_2}$ versions of Dirichlet Principle.

5^o. 34 pupils wrote a test. Nobody made more than 10 mistakes. Prove that at least 4 pupils made the same number of mistakes.

6^o. Are there at least 4 coins of the same value among any 25 coins of Latvia (i.e. 1 santīms, 2 sant., 5 sant., 10., sant., 20.sant., 50 sant., 1 lats (Ls), 2 Ls)?

7^o. There are 40 pupils in the class. Is there definitely such a month, when no less than 4 pupils of this class celebrate their birthdays?

8^o. There are 30 classes in the school, and 1000 pupils learn in them. Prove that in this school there is a class, where no less than 34 pupils learn.

9^o. In the library there are 1000 books. None of the books has more than 80 pages. Prove that in this library there are at least 13 books with the same number of pages.

10^o. 40 delegates from 13 regions participated at the conference. Prove that at least from one region no less than 4 delegates had come.

11^o. Prove: out of every 15 pupils you can choose three, born on the same day of a week.

12^o. a) Prove: out of any 10 natural numbers you can choose two beginning in the same digit.

b) Prove: out of any 11 natural numbers you can choose two ending in the same digit.

(Pay attention to the fact that the above mentioned quantity of numbers differs in both parts of the problem.)

In the previously discussed examples all the groups were reciprocally of equal worth, and no differences were noticed among them. Now, let us discuss an example in which all the groups are not of equal worth, and this difference has an essential influence upon the solution of the problem.

6. example. *Prove that among any 35 two-digit numbers (the first digit is not 0) you can find 3 with equal digital sums.*

Solution. Let us draw the following table: let us write along its horizontal side all the possible values of the first digit of the number, but along the vertical side - the possible values of the second digit. Let us write the sum of numbers of the column and the line into each square of the table (fig.1.). We have formed compliance between the numbers and the sums of their digits:

	1	2	3	4	5	6	7	8	9
0	1	2	3	...					
1	2	3	...						
2	3	...							
3	...								
4									
5									
6									...
7								...	16
8							...	16	17
9						...	16	17	18

Fig.1

You can independently make sure that all the two-digit numbers from 1 to 99 have been used in the table. In general, 18 different values of digital sum are possible (we can form 18 different groups). If we did as before, we would have no success, because we **can** distribute 35 numbers into 18 groups so, that there are no more than two numbers in any of them.

Nevertheless, we can point out that:

1) only one number (10 and 99) has digital sum 1 and 18 each,

2) only two numbers (11; 20 and 89, 98) have digital sums 2 and 17 each.

So, in these groups there cannot be more numbers irrespective of what 35 numbers we choose.

Let us presume that these 4 groups are filled to maximum - 6 numbers placed together into them. Then 29 numbers must be placed into the remaining 14 groups so, that there are no more than 2 numbers in each group. But according to Dirichlet Principle, it is definitely possible to find such a “hutch”, into which at least 3 numbers are placed. The assertion of the problem is proved.□

Problems for Independent Solution.

13^{o}. Prove that among any 78 three-digit numbers (the first digit is not 0) it is possible to find 4 numbers with equal digital sums.

14^{o}. There are 32 people on the train, who are over 70. Prove that among them it is possible to find either 2 persons over 80, or 4 persons of the same age and not over 80.

15*. In the leap-year there are 366 days. Prove that out of any 264 days you can choose either 4 whose sums of date digits are 4 for each day, or 26 days with equal sums of date digits.

16. 18 pupils wrote the test. One of them made 4 mistakes, the others - less. Prove that it is possible to find

5 pupils who made the same number of mistakes (maybe none).

II. ABOUT THE BUILDING OF "HUTCHES"

After a successful choice of groups in the initial problems of the previous chapter the way of further solution was clear. Besides it was clear from the problem itself, what groups must be discussed in each problem. However, it may happen that the groups ("hutches") themselves must be constructed somehow "cunningly", or some qualities of these groups must be proved before Dirichlet principle $\mathbf{D_1}$ or $\mathbf{D_2}$ is applied. We saw some examples at the end of the previous chapter.

In this chapter we shall discuss several examples, paying major attention to the method of construction of "hutches".

7. example. *A square table consists of 6 x 6 squares. In each square there is written "+1", "-1" or "0". Prove that, by calculating the sum of numbers written in each column, each line and each diagonal, two of the sums will be equal.*

Solution. "The objects" which we shall distribute will be the calculated sums. There are 14 of them. None of them exceeds 6 and is not smaller than (-6). So, only 13 different values are possible for the sums: -6; -5; -4; -3;
-2; -1; 0; 1; 2; ;3 ;4; 5; 6. As there are more sums than values (more "rabbits" than "hutches"), then from $\mathbf{D_1}$ it follows: two sums with identical values will turn up.□

8. example. *The Ignorant Boy*[+] *has 101 friends in the Blossom Town. It is known that in this town the inhabitants can have 10 different colours of eyes and 10 different colours of hair. Can you definitely assert that among the Ignorant Boy's friends there are two with the same colour of eyes and hair?*

Solution. Let us draw a table with the parameters 10 x 10 squares (fig. 2). Each column will serve one colour of hair, each line - one colour of eyes; in total there are 100 combinations of the colours of eyes and hair: each of them is depicted by one square ("hutch"). As there are more friends than squares, then at least in one "hutch" there get at least two friends; they have both the same colour of the eyes and of hair.□

9. example. *Round the round table there are sitting 20 persons: 11 men and 9 women. Prove that there are two men sitting opposite each other.*

Hair / Eyes	1	2	3	4	5	6	7	8	9	10
1										
2										
3										
4										
5										
6										
7										
8										
9										
10										

Fig. 2

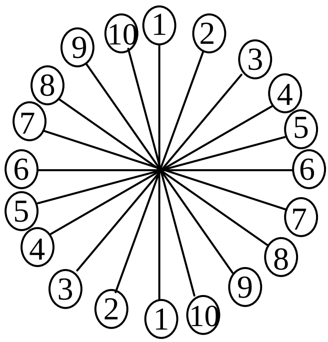

Fig. 3.

Solution. Let us make 10 "hutches": each of them consists of two places located opposite each other

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[+] A person in A. Nosev's book

(in the fig. 3 the places belonging to one "hutch" are marked with the same number from 1 to 10). As 11 men are distributed in 10 "hutches", then two of them have got into one "hutch"; both these two men are sitting opposite each other.□

10*. example. *There are 4 activists at some institution. They form commissions for different undertakings. Besides, the commission may consist of 1; 2; 3; 4 persons. No two commissions may completely coincide, and their number is 9. Prove that among the commissions there are two, which have not a common member.*

Solution. Let us mark the activists with A; B; C; D. Let us write down all the possible commissions; all the commissions, except one, will be combined in pairs.

1) ABCD	this commission has no pair
2) A;	BCD
3) B;	ACD
4) C;	ABD
5) D;	ABC
6) AB:	CD
7) AC;	BD
8) AD;	BC

Let us regard these 8 groups (7 pairs of commissions and one group consisting of 1 commission) as "hutches"; and we shall regard as "rabbits" the 9 commissions formed by the activists. In accordance with $\mathbf{D_1}$ two of these 9 commissions belong to one pair; but it is easy to see that the commissions combined in one pair have not a common member (the pairs are combined exactly after this principle).□

11[k]. example. *Pete has 100 small circles with numbers 1 to 100 written on them (a different number on each circle). The teacher told him to choose 4 circles and place them so to obtain a correct identity* ◯ + ◯ = ◯ + ◯.

Pete's circles had scattered on the floor and he had managed to gather only 21 circle, when he got the task. Will this quantity of circles be definitely enough for Pete to accomplish the teacher's task?

Solution. Let us think over, how many different pairs we can build of the circles gathered by Pete (we shall consider identical pairs, which differ only by the sequence of circles, as the same; for example ① ④ is the same pair as ④ ①).

Let us imagine that every two circles are linked with a string. As each of 21 circles has 20 ends of strings tied to them, then the total is $21 \cdot 20 = 420$ ends of strings; as each string has exactly 2 ends, there are $420 / 2 = 210$ strings. Therefore it is clear that Pete can make 210 pairs of circles altogether, using the gathered 21 circle. We shall regard these pairs as "rabbits" in the further judgement.

What can the sums of numbers, included in one pair, be like? The value of the smallest sum is $1 + 2 = 3$; the biggest value is $99 + 100 = 199$. So, there are possible $199 - 3 + 1 = 197$ different values of the sum: 3; 4; 5; …; 197; 198; 199. We shall regard these different values as "hutches"[++], i.e., there are at least two "rabbits" in some "hutch". It means that Pete has two pairs of circles,

[++] As there are more "rabbits" than "hutches", then in accordance with $\mathbf{D_1}$ in some "hutch" there is more than one "rabbit".

where the sums of included numbers are equal (identical); let us presume that A + B = C + D (the pairs are A, B and C, D).

None of the numbers is included in both pairs; really, if, for example, it were A = C, then from A + B = C + D it would follow also B = D and the pairs A, B and C, D would not be different.

Therefore, all four circles A, B, C, D are different, and so Pete can form the identity A + B = C + D out of them.□

Please note that after using Dirichlet principle in the solution of this example, the solution was not yet over - there followed an essential judgement on the difference of all four circles. We shall still find such situations several times.

In the following two examples the "hutches" and "rabbits" will be formed very ingeniously.

12[k]. example. *During 50 days in succession Jānītis (Johny) solved 79 problems in total. Besides, he solved at least one problem in a day. Prove that it is possible to find several days in succession (maybe only one), when he solved exactly 20 problems.*

Solution. Let us mark the number of problems solved by Jānītis during the first n days with a_n (n = 1; 2; 3; …; 50). Then $a_1 \geq 1$; $a_1 < a_2 < a_3 < \ldots < a_{49} < a_{50} = 79$ (*).

Let us look also at the numbers $a_1 + 20$; $a_2 + 20$; $a_3 + 20$; …; $a_{49} + 20$; $a_{50} + 20$. In accordance with (*) we get

$21 \leq a_1 + 20 < a_2 + 20 < \ldots < a_{49} + 20 < a_{50} + 20 = 99$ (*).

We shall regard as "rabbits" the numbers $a_1; a_2; \ldots; a_{50}; a_1+20; a_2+20; \ldots; a_{50}+20$; their possible values - as "hutches". There are 100 " rabbits". The possible values are from 1 to 99, therefore there are 99 of them - one less than the "rabbits"; we shall regard them (the values) as "hutches".

In accordance with **D₁**, at least two "rabbits" will get into one "hutch" (at least two numbers will have the same values). It is clear that there cannot be $a_i = a_j$ (if $i \neq j$) or $a_i + 20 = a_j + 20$ (if $i \neq j$); therefore, the identical numbers one is from group $\{a_1; a_2; \ldots; a_{50}\}$, but the other is from group $\{a_1 + 20; a_2 + 20; \ldots; a_{50} + 20\}$. If $a_i = a_j + 20$, then $a_i - a_j = 20$. In accordance with the definition of numbers $a_1; a_2; \ldots; a_{50}$ it means, that in $(j+1)$-th, $(j+2)$-th, …, i-th days altogether Jānītis has solved exactly 20 problems.□

13[k]. example. *The rectangle consists of 4 x 7 squares. Each square is coloured either white or black. Prove that it is possible to find such two lines and two columns, where all four squares, which are on their points of intersection, are coloured in the same colour.*

Solution. Let us prove more than is demanded in the problem: we shall prove that such two lines and two columns can already be found in the rectangle with parameters 3 x 7 squares (i.e., within the boundaries of **the part** of the rectangle mentioned in the problem).

A. Let us observe: in every column (it consists of 3 squares) it is possible to find 2 squares coloured in the same colour. It follows from Dirichlet principle **D_1**: colours are “hutches”, squares - “rabbits”.

B. Further, let us observe: if two columns are coloured alike, then it is possible to find the above mentioned 4 squares (in both columns you must take one and the same pair of squares coloured alike, see fig. 4.).

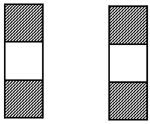


Fig. 4.

C. Let us remark that in total 8 different colourings of columns are possible (they all are shown in fig. 5.).

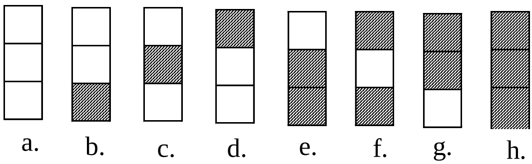


Fig. 5.

Further we shall discuss several situations.

C_1. Among 7 columns there is none, whose squares are coloured with the same colour (i.e., neither type a nor type h column). In this case each of these 7 columns belongs to one of the 6 types: a, b, c, d, e, f, g. In accordance with **D_1**, two columns belong to one and the same type, i.e., they are coloured alike.

In accordance with point B it is possible to find the necessary 4 squares.

C_2. Let us presume that some column of the 7 is coloured white (i.e., it belongs to type a). If at least one of the 6 remaining columns belongs to any of the types a, b, c, d, then, looking at this column and the white one, it is possible to find the 4 squares located as required (see fig. 6.).

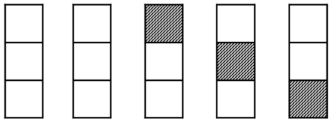


Fig. 6.

Now it remains to discuss the situation, when none of the 6 remaining columns is of a, b, c, d type; then each of these 6 columns belongs to one of the 4 types e, f, g, h. In accordance with Dirichlet principle $\mathbf{D_1}$, at least two of the columns belong to the same type. Applying the judgement of B point to these two columns, we get the required 4 black squares.

C_3. Some of 7 columns are all over coloured black. We analyse this situation like **C_2**. All the situations are analysed, the problem is solved.□

Commentary. The solution of this problem is very didactic. First, note that Dirichlet principle was used "in two stages": now, to state the existence of alike coloured columns, now further to state the existence of alike coloured squares inside these columns.

Second, several situations were sorted, and in each of them Dirichlet principle was applied in a different situation (6 “hutches” and 7 “rabbits”, 4 “hutches” and 6 “rabbits”).

Third, surprise could be caused by the fact that we made narrower the rectangle mentioned at the beginning (from 4 x 7 to 3 x 7): it may seem that we have deliberately passed over to a worse situation. Really, you can find the necessary thing in a bigger rectangle as well as in a smaller one!

Test it yourself, that in the rectangle 4 x 7 you would not be able to make this kind of statement: columns consisting of 4 squares can be coloured in 16 ways. So, remaining in the 4 x 7 rectangle, you would have to look for some other way of solution.

The situation, when it is easier to solve an as-if more difficult problem, is very often met in mathematics (and also in life).

In the following problem come next still other judgements after the application of Dirichlet principle.

14. example. *In the artists’ study 36 sculptures have been made and their mass is 490 kg, 495 kg, 500 kg, ..., 665 kg.*

Is it possible to transport all these sculptures in 7 trucks, if the carrying capacity of each truck is 3 tons, each truck may make only one run and you must not overload the truck?

Solution. The total mass of sculptures is 20790 kg < < 21000 kg = $7 \cdot 3000$ kg. So, if instead of sculptures there was, for example, sand which can be emptied in trucks, then it could be transported according to the rules of the problem.

But you must not divide the sculptures in parts. Therefore the fact that the total mass of sculptures is

smaller than the total carrying capacity of trucks does not prove that sculptures can be transported as indicated in the problem. Let us observe that $36 = 7 \cdot 5 + 1$ sculptures must be distributed for 7 trucks. In result, no less than 6 sculptures must be loaded on one truck. But even total mass of 6 lightest sculptures is $490 + 495 + 500 + 505 +$
$+ 510 + 515 = 3015$ kg > 3000 kg, so it is bigger than the mass allowed for one truck.

It means that the demands of the problem cannot be fulfilled.□

Further there are given the **problems for independent solution**. You can use $\mathbf{D_1}$ and $\mathbf{D_2}$ versions of Dirichlet principle in all these problems. The main thing is - choose "hutches" and "rabbits" in the appropriate way.

16°. A square table consists of 10 x 10 squares. You write "+ 1", "- 1" or "0" in each square. Prove that, when calculating the sums of numbers written in each line, each column and each diagonal, two identical sums will turn up among them.

17. Are there existing 11 positive two digit numbers, which form an increasing arithmetical progression and whose digit sums also form an increasing arithmetical progression in the same order?

18. There are 6 active pupils in the class. They have formed 30 commissions. It is known that each 2 commissions have at least one common member. Prove that it is possible to form one more commission so that this quality remains. Besides, the membership of this new commission must not coincide with any of the existing commissions.

19. There are 25 pupils in the classroom. Each of them has blue, brown or grey eyes. It is known that among every 3 children there are at least two of the same age. Prove that it is possible to find either 3 boys or 3 girls, who are of the same age and with the same colour of eyes.

20*. Pēterītis (Pete) has 69 boxes. Different natural numbers not exceeding 100 are written on the boxes (one number on each box). Prove that it is possible to place 4 boxes so, that the equivalence $\square + \square + \square = \square$ is fulfilled.

21^o. Jānītis (Johny) was preparing for mathematical Olympiad for 11 weeks. Every day he solved at least one problem, but, no to overwork himself, he solved no more than 12 problems during any of the weeks. Prove that it is possible to find several days in succession, when Jānītis solved exactly 21 problem altogether.

22*. A rectangle consists of 5 x 41 squares. Each square is painted in one of two colours. Prove that it is possible to find such three lines and three columns, where all the squares on their points of intersection are painted in the same colour.

23*. There are 65 deputies in the parliament. In none of the parliament commissions there work all the deputies. It is known that every two deputies work together **exactly** in one commission. Prove that there is such a deputy, who is engaged in at least nine commissions.

24^k. Different two-digit numbers are written on 10 cards (one number on each card). Prove that it is possible to make simultaneously two small heaps so, that the sums of these heaps are the same (it is allowed not to place some cards in any of the two heaps).

25. The apexes of a regular 100-gon in some order are numerated with natural numbers from 1 to 100 (each apex with a different number). We calculate the difference of numbers of the ends for each side (the smaller number is subtracted from the bigger). Prove that for at least two sides these differences will turn out the same.

26. On one Sunday 7 friends decided to attend shows in 9 cinemas beginning at 9^{00}, 10^{00}, 11^{00}, …, 17^{00}, 18^{00}, 19^{00}. Two friends watched each of the shows at the same cinema, the other 5 friends - at some other cinema. In the evening it turned out that each of the friends had been at all 9 cinemas. Prove that in each of the 9 cinemas none of the friends had attended at least one show.

27^k. 101 different natural numbers are written in a line. Prove that it is possible to cross out 90 of them in such a way, that the remaining 11 numbers would be put either in an increasing or diminishing order (i.e., either each following of the remaining numbers exceeds the preceding, or each following of the remaining numbers is smaller than the preceding).

28*. The table consists of 10 x 10 squares. In each square there is written a number (all the numbers are different). It is known that in each line the numbers are increasing from left to right. Jānis rearranged the numbers in each column so, that the numbers increase downwards from the top in each column (the numbers **were not moved** from one column to the other). Prove that after the rearrangement the numbers in each line are still increasing from left to right.

29. Between numbers 1 and 100 there are chosen 11 different numbers (not definitely whole numbers!). Prove that among them it is possible to find such two numbers, whose ratio is grater than 1, but it does not exceed 1,6.

30. In each of the apexes of the 28-gon there must be written one of the letters A; B; C; D: E; F; G; H so, that all its sides were marked with different pairs of letters. (Pairs XY and YX are considered identical; sides, marked with one and the same letter at both ends, are not allowed). Can it be done?

III. PROBLEMS ABOUT THE DIVISIBILITY OF NUMBERS

The problems about the divisibility of numbers form a large class of problems, where it is possible to apply Dirichlet principle.

III.1. PROBLEMS CONNECTED WITH THE CONCEPT OF REMAINDER

We shall remind of the main facts which must be remembered to understand the examples analysed in this chapter.

If two whole numbers m and n are given, moreover, $m > 0$ (i.e., m - natural number), then you can divide n by m “with remainder”. It means that such whole numbers q and r are found, where

$$n = q \cdot m + r, \text{ besides } 0 \le r < m.$$

Number q is called quotient, but r - remainder. If $r = 0$, then it is said that n is divisible by m without remainder (or simply, that n is divisible by m).

If numbers n and m are given, then the quotient q and remainder r are definitely unique. It is important to remember that, dividing by natural number m, the remainder can adopt only values $0; 1; 2; \ldots; m-1$, i.e., m different values.

Eamples. *Dividing 12 by 7, the quotient is 1, the remainder 5, because 12 = 1 · 7 + 5. Dividing 12 by 6, the quotient is 2, the remainder is 0, because*

12 = 2 · 6 + 0. Dividing -4 by 2, the quotient is -2, the remainder is 0, because -4 = -2 · 2 + 0. Dividing -13 by 7, the quotient is -2, the remainder is 1, because -13 = -2 · 7 + 1.

The following two theorems will be used.

Theorem about the division of difference (TDD theorem)

Let us presume that a, b and n are whole numbers, moreover, n > 0. Difference a- b is divisible by n if and only if a and b give the same remainders, when divided by n.

▼ Let us prove this theorem. In the proof we must discuss two cases.

1) Let us presume that a and b give the same remainders, when divided by n. Let us mark the common value of these remainders with r. Then $a = q_1 \cdot n + r$, $b = q_2 \cdot n + r$. By subtracting the second equivalence from the first one, we get $a - b = (q_1 - q_2) \cdot n$ (the remainder n reduced). The obtained equivalence also shows that a - b is divisible by n (the quotient is $q_1 - q_2$).

2) Let us presume that a - b is divisible by n and number b gives the remainder r, when divided by n. It means that $b = q_1 \cdot n + r$ and $a - b = q_2 \cdot n$ (q_1 and q_2 - some whole numbers). Then $a = q_2 \cdot n + b = q_2 \cdot n + q_1 \cdot n + r = (q_1 + q_2) \cdot n + r$. From the identity $a = (q_1 + q_2) \cdot n + r$ we see that, dividing a by r, we get the quotient $q_1 + q_2$ and the remainder r; so really a and b give the same remainders, when divided by n.

The **TDD** theorem is proved.▲

The **TDD** theorem is often used together with the Dirichlet principle version, which we shall call $\mathbf{D_3}$.

Theorem D_3.

If n objects are distributed into n groups so that in none of the groups there is more than one object, then in each group there is exactly one object.

If $\mathbf{D_1}$, $\mathbf{D_2}$ or $\mathbf{D_3}$ is used together with **TDD**, then usually (but not always!) "hutches" are **the possible values of remainder**, but "rabbits" - **remainders themselves**, which are obtained by dividing some numbers by the respective n.

15. example. *Prove that from any 8 natural numbers you can choose two so, that their difference is divisible by 7.*

Solution. We take arbitrary natural numbers $a_1, a_2, \dots, a_8$. Dividing them by 7, we shall obtain the remainders, which we shall mark $b_1, b_2, \dots, b_8$. Dividing by 7, the remainders can be only $0, 1, \dots, 6$ - seven different values altogether. It follows from Dirichlet principle $\mathbf{D_1}$, that among the remainders $b_1, b_2, \dots, b_8$ there are at least two identical ones.

According to **TDD** theorem those two numbers, whose remainders, divided by 7, are identical, meet the terms of the problem.□

31°. Does the assertion of the problem remain valid if 8 is substituted by an arbitrary natural number n, but 7 - by n - 1?

Reference: please, especially discuss the situation n = 1!

16. example. *It is given that a, b, c, d are arbitrary natural numbers. Prove that* $(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$ *is divisible by 12.*

Solution. We must prove that that the product under discussion (we shall mark it R!) is divisible by 3 and 4. Let us divide each of the numbers a, b, c, d by 3 with remainder. We shall obtain four remainders. They can have only three different values: 0, 1 and 2. In accordance with $\mathbf{D_1}$, the values of two remainders will be equal.The difference of the respective numbers will be divisible by 3 in accordance with **TDD** theorem. Therefore the whole product R will be divisible by 3.

Let us try to decide in the same way about the division by 4. We shall divide each of the numbers a, b, c, d by 4. We shall get 4 remainders. Unfortunately, as 4 different values of remainders are possible (0, 1, 2 and 3), we cannot assert that among these values will be two identical.

Therefore we shall do otherwise: we shall discuss two possible situations.

1) Two of the obtained remainders are equal. Then the difference of the respective numbers is divisible by 4, therefore the product R is divisible by 4.

2) No two of the obtained remainders are equal. Then, in accordance with $\mathbf{D_3}$, exactly one of them is 0, exactly one is 1, exactly one is 2 and exactly one is 3. But the difference of those numbers, which give the remainders 0 and 2, is divisible by 2 (really,

$(4q_1+2)-(4q_2+0)=2(2q_1+1-2q_2)$, so is divisible by 2). In alike manner also the difference of those numbers, which give the remainders 1 and 3, is divisible by 2 (the demonstration is similar, do it independently).

So two brackets contained by R are divisible by 2; therefore R is divisible by 4.

The problem is solved. □

17*. example. *100 whole numbers are written in a line. Prove that it is possible to choose some numbers written in succession (maybe one number), whose sum is divisible by 100.*

Solution. We have already gained certain skill, and our intuition tells us that this time the role of "hutches" is taken by the values of remainders, which can arise when some number is divided by 100: $0, 1, 2, \ldots, 98, 99$; there are 100 such values. But what should we choose for the role of "rabbits"?

Let us try to group the given numbers $A_1, A_2, \ldots, A_{100}$ depending on the remainder they give when divided by 100. If among these numbers there are two, which give equal remainders, when divided by 100, then … nothing helps, because it does not at all mean that the sum of these numbers will be divisible by 100. Besides, there is not any guarantee that these numbers are written in succession in the given line.

So, such choice of "rabbits" is not successful, and we must act in some other way.

Let us discuss 100 sums (we shall name A_1 a sum for convenience sake):

$$S_1 = A_1,$$

$$S_2 = A_1 + A_2,$$
$$S_3 = A_1 + A_2 + A_3,$$
$$\dots$$
$$S_{100} = A_1 + A_2 + \dots + A_{100},$$
and let us discuss the remainders given by each sum, when divided by 100. So, 100 different values of remainders are possible.

If any of the sums gives the remainder 0, then the numbers building this sum are those we were looking for in problem.

However, if none of the sums is divisible by 100, then according to **D₁**, among the discussed 100 sums there are at least two (we shall mark them S_i and S_j; S_i holds less items than S_j), which give equal remainders when divided by 100. Their difference $S_j - S_i$ by the **TDD** theorem is divisible by 100. It is apparent that this difference consists of the numbers A_{i+1}, A_{i+2}, …, A_{j-1}, A_j taken from the given chain by turn in succession.

These numbers meet the term of the problem.□

18*. example. *Prove that there exists a natural number, whose last four digits are 1990 and which is divisible by 1991.*

Solution. Let us discuss 1991 number:

$$A_1 = 1990,$$
$$A_2 = 19901990,$$
$$A_3 = 199019901990,$$
$$\dots$$
$$A_{1991} = \underbrace{1990\dots1990}_{1991}.$$

Let us divide each of them by number 1991 with remainder. We can get 1991 different values of remainders: $0; 1; 2; \ldots; 1990$. Further we separate two possibilities.

1) all the obtained remainders are different. Then, according to **D₃**, among them there is also the remainder 0, i.e., there exists number A_1, which is divisible by 1991. We can take it as the necessary one.

2) Two numbers (e.g., A_i and A_j, $i > j$) give identical remainders when divided by 1991. Then the difference of these numbers is divisible by 1991 in accordance with **TDD** theorem. But this difference is

$$A_i - A_j = \underbrace{19001900\ldots1900}_{i-j \text{ times}}\underbrace{0000\,0000\ldots0000}_{j \text{ times}}.$$

To write a zero at the end of the number means to multiply it by 10, i.e., by 2 and by 5. It is clear that zeros at the end of the number have no effect on divisibility by 1991, because 1991 does not contain either 2 or 5 as divisors.

It means that number $\underbrace{19001900\ldots1900}_{i-j \text{ times}}$ is divisible by 1991. We can take this number as the one we were looking for.□

<u>19. example.</u> *Is it possible to find such a degree of number 3 (the index - a natural number), which ends on in digits 0001?*

<u>Solution</u>. Yes, it is possible. We shall prove it. Let us discuss the groups of the last four digits of numbers $3^1, 3^2, 3^3, \ldots, 3^{10001}$. Four digits can build up 10 000 different groups: 0000, 0001, 0002, …, 9998, 9999. As we are discussing more numbers than the different

groups, then for two numbers these groups will be identical. They will "shorten" by subtraction, and the difference of the respective numbers will end in 0000 or will be divisible by 10 000. We get $3^n - 3^m = 10000 \cdot q$ (q - quotient), $n > m$. We shall write down this identity as $3^m \cdot (3^{n-m} - 1) = 10\,000 \cdot q$. We see that $3^m (3^{n-m} - 1)$ is divisible by 10 000.

As 3^m will not even reduce with 10 000 (3^m consists of threes, but 10000 = 2 · 2 · 2 · 2 · 5 · 5 · 5 · 5), then in reality $3^{n-m} - 1$ is divisible by 10 000 or $3^{n-m} - 1 = \ldots 0000$. From there it follows that $3^{n-m} = \ldots 0001$, which we had to prove. The problem is solved.□

Commentary.

1. Please note that the group of the last 4 digits of the number forms the remainder, which is obtained by dividing the given number by 10 000. So, in the discussed solution the role of "rabbits" again was taken by the remainders; and the **TDD** theorem was used though not mentioned.

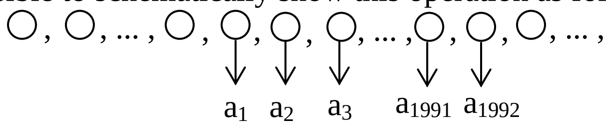
2. In the solution it would have been enough to discuss also only the powers $3^1, 3^2, \ldots, 3^{10000}$ and to observe: if among them there is a power, whose 4 last digits are 0001, then the necessary result is already obtained; if such a power is missing, then there are no more then 9999 chains of last four digits per 10 000 powers, and for two powers these chains are identical. Further on we are judging like in the preceding solution.

3. In reality the discussion of a still smaller quantity of powers would have been enough, because no power of three can end in, for example, even number or number 5; therefore there are much less than 10 000 of the possible chains ("hutches") of last 4 digits.

20*. example. *Prove that from each infinite digital chain it is possible to choose some digits written in a succession so, that the number, consisting of these digits written in succession in the chain, would be divisible by 1991.*

Solution. If in the given digital chain at least one element is 0, then the solution of the problem is trivial, because by choosing this digit we get the number 0, which is divisible by 1991. Therefore we shall presume that no member of the given infinite digital chain is equal with zero.

Let us choose some element of the given chain and "cut out" of the chain a fragment which begins in this chosen element and whose length is 1992 digits. We shall numerate the elements of the obtained fragment without changing their succession, with numbers $1, 2, \ldots, 1992$ and mark these elements with $a_1, a_2, \ldots, a_{1992}$. It is possible to schematically show this operation as follows:



where the elements of the given chain are shown with circles.

Let us look at the chains of digits:

$$a_1, a_2, a_3, \ldots, a_{1991}, a_{1992};$$

$$a_2, a_3, \ldots, a_{1991}, a_{1992};$$
$$a_3, \ldots, a_{1991}, a_{1992};$$
$$\ldots\ldots$$
$$a_{1991}, a_{1992};$$
$$a_{1992}.$$

(Every next chain is obtained by dismissing the first digit from the left side in the preceding chain).

To each such digital chain there corresponds a whole number, which is obtained by wiping off all the commas (for example, if some of the chains is 5, 7, 3, 3, 1, then number 57331 corresponds to it). We have obtained 1992 numbers. They all are different. Let us discuss the remainders, which arise by dividing by 1991 each of these numbers. Only 1991 different values of remainders are possible. But as there are 1992 numbers, then at least two remainders are identical. Let us presume that the number of digits in these numbers is n and k (n > k), and mark these numbers, by S_n and S_k. As we obtained the numbers with a smaller number of digits from the numbers with a bigger number of digits by dismissing one or several first digits, then the last k digits of the numbers S_n and S_k coincide. By subtracting the smaller number from the bigger we obtain the result, where at beginning there are the digits of the given chain taken in succession (the digits which are not yet included in number S_k), but k zeros follow after them. If we mark with a the above mentioned number, formed by the digits taken in succession from the given chain, then the result is $a\underbrace{000\ldots000}_{k}$ or $a\cdot 10^k$. In accordance with **TDD** theorem this difference is divisible by 1991. But 10^k is not even reducing by 1991, therefore a is divisible by 1991. The necessary digital group of the chain given at the beginning has been found.□

<u>21. example.</u> *Prove that among every 3 natural numbers it is possible to find exactly two numbers, whose sum is divisible by 2.*

<u>Solution</u>. Let us distribute all the numbers ("rabbits") into two "hutches" depending on their being either even or odd numbers. Two (or even all 3) numbers will get into some "hutch". We choose two numbers from one "hutch". As the sum of both two even and two odd numbers is an even number (so, it is divisible by 2), the problem is solved.□

<u>22. example.</u> *Prove that among every 5 natural numbers it is possible to find exactly three numbers, whose sum is divisible by 3.*

<u>Solution</u>. If there were not the demand for exactly three items, the problem could be solved just like the 17. example; besides there would not be needed 5 numbers, but 3 numbers would be enough. However, the method of the 17. example does not guarantee anything as regards the number of items. Therefore we must do otherwise.

We shall distribute all the numbers ("rabbits") into three "hutches" depending on the remainder, which each of the numbers gives when divided by 3 (as it is known, only the remainders 0, 1 and 2 are possible). Let us discuss two possibilities.

1) In every "hutch" there is at least one number. Let us take one number from every "hutch"; let us mark them with $3a$, $3b + 1$ and $3c + 2$. Then the sum of

these 3 numbers $S = 3a + (3b+1) + (3c+2) =$
$= 3(a+b+c+1)$ is divisible by 3.

2) There is such a "hutch", where there isn't any number. It means that all five numbers are distributed into no more than two "hutches"; in accordance with **D_2** in some "hutch" there are at least 3 numbers (if in every "hutch" there were no more than two numbers, then in total there would not be more than four numbers). We take 3 numbers from one "hutch". We assert that they are valid as the numbers we were looking for.

Really, if they give the remainder r (one and the same!) when divided by 3, then we can mark them as $3a + r$, $3b + r$ and $3c + r$ and their sum is $3(a+b+c+r)$, i.e., divisible by 3. The necessary fact is proved.□

23. example. *Prove that among every 52 natural numbers you can find two, whose sum or difference is divisible by 100.*

Solution. Let us form 51 "hutch".

In the 1. "hutch" we shall put numbers whose remainders, divided by 100, are 1 or 99;

In the 2. "hutch" we shall put numbers whose remainders, divided by 100, are 2 or 98;

In the 3. "hutch" we shall put numbers whose remainders, divided by 100, are 3 or 97;

…

In the 49. "hutch" we shall put numbers whose remainders, divided by 100, are 49 or 51;

In the 50. “hutch” we shall put numbers whose remainders, divided by 100, are 50;

In the 51. “hutch” we shall put numbers whose remainders, divided by 100, are 0.

As there are more numbers than “hutches”, then two numbers will get into one hutch”. If two numbers get into the “hutch” 50 or 51, then both their sum and their difference is divisible by 100. If two numbers get into one of the first 49 “hutches”, then:

- if their remainders, divided by 100, are the same, their difference is divisible by 100;
- if their remainders, divided by 100, are different, their sum is divisible by 100 (just because of this property the “hutches” were built in the chosen way).

The problem is solved.□

Note. The reader can easily verify that it is not possible to substitute number 52 by 51: if, for example, numbers 50; 51; 52; …; 98; 99; 100 were chosen, then neither the sum of any two numbers nor their difference would be divisible by 100.

24. example. *The squares of 5 natural numbers are given. Prove that it is possible to choose from them such two numbers, whose difference is divisible by 7.*

Commentary. Let us take into consideration: if the given numbers were not squares but arbitrary natural numbers, the assertion of the problem would not be correct; it is sufficient to discuss just numbers 1; 2; 3; 4; 5. The essential difference is in this fact: although natural numbers, divided by 7, can give 7 different remainders, their squares have less of possible remainders. This property shows itself not only when the division by 7 is discussed, but also the division by other numbers.

Now we shall look at two different solutions.

<u>1. solution</u>. Let us check up, what remainders can be given by the square of the number, when divided by 7. We shall also discuss the situations depending on what remainder is given by the number itself, when divided by 7.

If $n = 7q + 0$, then $n^2 = 49q^2$; the remainder is 0.

If $n = 7q + 1$, then $n^2 = 49q^2 + 14q + 1$; the remainder is 1;

If $n = 7q + 2$, then $n^2 = 49q^2 + 28q + 4$; the remainder is 4;

If $n = 7q + 3$, then $n^2 = 49q^2 + 42q + 9$; the remainder is 2 (because $9 = 7 + 2$);

If $n = 7q + 4$, then $n^2 = 49q^2 + 56q + 16$; the remainder is 2 (because $16 = 14 + 2$);

If $n = 7q + 5$, then $n^2 = 49q^2 + 70q + 25$; the remainder is 4 (because $25 = 21 + 4$);

If $n = 7q + 6$, then $n^2 = 49q^2 + 84q + 36$; the remainder is 1 (because $36 = 35 + 1$).

We see that only 4 different values are possible for the remainder of the square. In accordance with $\mathbf{D_1}$ two of the discussed squares give equal remainders, when divided by 7; in accordance with **TDD** their difference is divisible by 7. The problem is solved.□

<u>2. solution</u>. Let us take into consideration that $a^2 - b^2 = (a+b)(a-b)$. So, it would be sufficient for us to prove that among every 5 natural numbers it is possible to find two, either whose sum or difference is

divisible by 7. It can be done like in the solution of the 23. example, discussing the following 4 “hutches”:

I - numbers with remainder 0, when divided by 7;

II - numbers with remainder 1 or 6, when divided by 7;

III - numbers with remainder 2 or 5, when divided by 7;

IV - numbers with remainder 3 or 4, when divided by 7.

Please fill in the details yourselves.□

Problems for Independent Solution.

31^o. Prove that from any 11 natural numbers it is possible to choose two so, that their difference would be divisible by 10.

32^o. It is given that a, b, c, d, e are natural numbers. Prove that the product of 10 brackets $(a-b)(a-c)(a-d)(a-e)(b-c)(b-d)(b-e)(c-d)(c-e)(d-e)$ is divisible by 288.

33^o. Prove that there exists a number, whose decimal record consist of only ones, and that it is divisible by 1977.

34^o. Prove that there exists a number, whose decimal record consists of only ones and zeros, and it is divisible by 1995.

35^o. Does there exist such a power of number 7 with a natural index, which ends in the digits 001?

36^o. 17 whole numbers are written in a line. Prove that it is possible to choose some numbers written in succession (maybe only one number), whose sum is divisible by 17.

37^o. Prove that among every 17 natural numbers it is possible to find exactly five, whose sum is divisible by 5.

38^o. Prove that among every 37 natural numbers it is possible to find exactly seven, whose sum is divisible by 7.

39. Generalize the results of the problems 37 and 38!

40^o. Prove that among any 502 natural numbers it is possible to find two, whose sum or difference is divisible by 1000.

41^o. Prove that among any 3 natural numbers it is possible to find two, whose difference of squares is divisible by 4.

42^o. Prove that among any 4 natural numbers it is possible to find two, whose difference of squares is divisible by 8.

43. Prove that among any 4 natural numbers it is possible to find two, whose difference of cubes is divisible by 9.

44. Prove: modifying a rational number $\frac{m}{n}$ in an infinite decimal fraction, the length of the period does not exceed n - 1.

45. Prove: if the greatest common measure (G.C.M.) of numbers a and b is 1, it is possible to find such natural number n, that $a \cdot n + 7$ is divisible by b.

46*. A Fibonacci chain of numbers is called such a chain, whose first two members are 1 and 2 (exactly in this order), but every following member is obtained by adding up both previous ones: $1; 2; 3; 5; 8; 13; 21; 34; \ldots$

Prove that in this chain there is a number divisible by 101.

47. Numbers from 1 to 101 were written in a chain in some order. After that the number of its place in the chain was added up to each number (1 was added up to the first number, 2 - to the second etc.). Prove that the product of all the obtained sums is an even number.

48^k. In an increasing arithmetic progression, which consists of 12 natural numbers, the difference does not exceed 1995. Prove that not all elements of this progression are prime numbers.

III.2. PROBLEMS CONNECTED WITH THE DECOMPOSITION OF NUMBER INTO THE PRODUCT OF PRIME NUMBERS

Let us remind of the main facts which can be used in solving the problems of this item.

1. If a and b are whole numbers and at least one of them is not 0, then the biggest natural number, by which both a and b are divisible, is called greatest common measure of a and b. It is denoted by G.C.M. (a, b).

For example. *G.C.M. (8, 6) = 2, G.C.M. (8, 11)* = 1, *G.C.M. (-4, -2)* = 2, *G.C.M. (6, 0)* = 6,

2. Each natural number $n > 1$ can be decomposed into the product of prime numbers. Besides, for each number various decompositions may differ only by the order of multipliers, but not by multipliers themselves, or the number of their repetitions. This fact is called **the fundamental theorem of arithmetics**.

For example. $8 = 2 \cdot 2 \cdot 2 = 2^3$; $42 = 2 \cdot 3 \cdot 7$; $100 = 2^2 \cdot 5^2$ *etc.*

(**Remember:** number 1 according to the definition is not a prime number!)

3. Two numbers are called **reciprocal prime numbers** if their greatest common measure is 1. By representing the reciprocal prime numbers into multipliers, no prime turns up in both representations. Also the other way round: if representations of two numbers into prime multipliers do not contain any common prime number, then their greatest common measure is 1.

For example. *As 88 = $2^3 \cdot 11$ and 121 = 11^2 (both contain the prime multiplier 11, so they are divisible by 11), then G.C.M. (88, 121) $\neq 1$; so 88 and 121 are not reciprocal prime numbers.*

As 216 = $2^3 \cdot 3^3$ and 77 = $7 \cdot 11$ do not contain common prime multipliers, then G.C.M. (216, 77) = 1.

Now we shall show how the complete or partial decomposition of a natural number into the product of prime numbers can be used by applying Dirichlet principle.

25*. example. *From numbers 1; 2; 3; ...; 200 exactly 101 number is chosen freely. Prove that among the chosen numbers it is possible to find two such numbers, from which one number is divisible by the other.*

Solution. We shall write each of the chosen numbers x in the form $x = n \cdot 2^k$, where n is an odd number (for example, $30 = 15 \cdot 2^1$, $31 = 31 \cdot 2^0$, $32 = 1 \cdot 2^5$, $199 = 199 \cdot 2^0$). Apparently all the odd multipliers n are smaller than 200. Among the numbers from 1 to 200 there are only 100 different odd numbers. Therefore among 101 chosen numbers in accordance with $\mathbf{D_1}$ it will be possible to find two such numbers, whose odd multipliers are equal. Let us presume that they are numbers $A = 2^n \cdot y$ and $B = 2^m \cdot y$. As $A \neq B$, then $n \neq m$. Let us presume that n > m, then $\frac{A}{B} = 2^{n-m}$. As n > m, then n - m is a natural number and 2^{n-m} - also a natural number. So A is divisible by B. The problem is solved.□

Commentary. Let us take into consideration that in the solution we didn't need to discuss complete representations of the numbers into multipliers - it was sufficient to distinguish the powers of only one prime multiplier 2 among the other prime multipliers.

26*. example. *From numbers 1; 2; 3; ...; 200 exactly 101 number is chosen freely. Prove that at least two of the chosen numbers are reciprocal prime numbers.*

Solution. We make 100 "hutches":

1. "hutch" - numbers 1 and 2;
2. "hutch" - numbers 3 and 4;

…

100. "hutch" - numbers 199 and 200.

As in 100 "hutches" 101 number is distributed, then two of them have got into one "hutch". These numbers - let us mark them a and b - differ from each other by 1. We assert that G.C.M. (a, b) = 1. Really, if G.C.M. (a, b) = x, then both a and b are divisible by x. Then the difference of a and b is also divisible by x; but this difference is 1. The only natural number by which 1 is divisible is 1. Therefore x = 1. The necessary fact is proved.□

27k. example. *Given are 5 natural numbers which exceed 1 and do not exceed 120. It is known that none of them is a prime number. Prove that it is possible to find among them two such numbers, whose greatest common measure exceeds 1.*

Solution. If $1 < n \leq 120$ and n is not a prime number, then we can express n = a · b, where a > 1, b > 1, a and b are natural numbers. We assert that that either a < 11 or b < 11. (Really, if on the contrary we presume that $a \geq 11$, $b \geq 11$, then we would obtain $a \cdot b \geq 121$, which contradicts with the given $a \cdot b \leq 120$). We can presume that a < 11. As a > 1, then a can be represented as the product of prime numbers (maybe as such a product, which consists of only one multiplier, if a itself is a prime number). These prime multipliers do not exceed 11. We obtain that a (and with it also n) is divisible by some prime number, which does not exceed 11. Let us observe that prime numbers not exceeding 11 are 2; 3; 5; 7. Let us form 4 "hutches" and write on them "2", "3", "5" and "7" respectively. We shall place each of the 5 numbers, mentioned in the problem, into the "hutch", where **the smallest** prime multiplier of this number is written. (For example, we place 60 into the "hutch" marked "2", because 60 = 2 · 3 · 5; we in our turn place 49 in the "hutch" marked "7", because 49 = 7 · 7). As there are more numbers than "hutches", then two numbers will turn up, which get into one "hutch". The G.C.M. of these numbers exceeds 1 (they both are divisible by the prime number, which is written on their common "hutch").□

28^k. example. *4 natural numbers are given. None of them is divisible by some prime number exceeding 4. Prove that among these numbers it is possible to find some (maybe only one), whose product is the square of some whole number. (If we choose only one number, then we regard this number itself as the product consisting of one multiplier.)*

Solution. If any of the given numbers is 1, then it itself is a square; we can choose it. If all four given numbers exceed 1, then each of them can be written down in the form $2^a \cdot 3^b$, where a, b are whole numbers, $a \geq 0$ and $b \geq 0$; really, in the representation of the number into product of prime numbers only such prime numbers may turn up, which do not exceed 4, i.e., nothing else than 2 and 3 (maybe also only one of these prime numbers).

Every number belongs to one of the 4 groups:

A	a -even number,	b - even number;
B	a -even number,	b - odd number;
C	a -odd number,	b - even number;
D	a -even number,	b - odd number.

Further we distinguish 2 situations.

1) There are not two numbers belonging to one of the groups mentioned above.

Then, according to $\mathbf{D_3}$, each of these groups (so, also the first group) possess exactly one number. We discuss the number belonging to the first group; it can be written down in the form $2^{2c} \cdot 2^{2d}$, where $a = 2c$, $b = 2d$, c and d are natural numbers or 0. It is clear that this number can be written down also in the form $(2^c \cdot 3^d)^2$, i.e., it is the square of the natural number $2^c \cdot 3^d$.

2) Two numbers (we shall mark them $2^m \cdot 3^n$ and $2^k \cdot 3^t$) are belonging to the same group.

In accordance with the formation of groups the numbers m and k have identical parity (they both are either even numbers or both - odd numbers). In the same way the n and t parities are identical. Therefore, m + k and n + t both are even numbers; we can mark $m+k=2u$, $n+t=2v$, where u and v are natural numbers or 0.

Then it remains to observe that the product $(2^m \cdot 3^n)\cdot(2^k \cdot 3^t) = 2^{m+k} \cdot 3^{n+t} = 2^{2u} \cdot 3^{2v} =$
$= (2^u \cdot 3^v)^2$ is the square of the natural number $2^u \cdot 3^v$. The problem is solved.□

Problems for Independent Solution.

49°. Out of numbers 1; 2; 3; …; 2n - 1; 2n we choose, without limitations, n + 1 number. Prove that one of the chosen numbers is divisible by some other.

50. Out of odd numbers 1; 3; 5; 7; …; 99; 101 we choose 35 numbers. Prove that one of the chosen numbers is divisible by some other.

51°. Out of numbers 1; 2; 3; …; 2n - 1; 2n we choose n + 1 number. Prove that among the chosen numbers it is possible to find two, whose greatest common measure is 1.

52°. Given are n natural numbers, which all exceed 1 and are smaller than $(2n-1)^2$. It is known that every two numbers have 1 as their G.C.M. Prove that at least one of them is a prime number.

53. Solve the problem mentioned in the 28th example, if given are not 4 but only 3 numbers with the above mentioned properties.

54. Given are 16 natural numbers written in a row, which all exceed 1. They all together have only 4 different prime multipliers. Prove that it is possible to choose some of them (maybe only one), whose product is the square of a natural number and which are written in succession in the given row.

55[k]. Given are 1995 natural numbers written in a row, which all exceed 1. None of them is divisible by any prime number exceeding 28. Prove that it is possible to choose some of these numbers, whose product is the fourth power of a natural number.

IV. DIRICHLET PRINCIPLE IN THE PROBLEMS ABOUT FINDING THE LARGEST AND THE SMALLEST VALUE

We often see problems, where it is demanded to find the possibly biggest or possibly smallest value of some magnitude. More often we find mistakes of logical nature in the solutions of such problems.

Let us discuss an example.

29. example. *What biggest quantity of vertexes of a 12 -gon may be on one straight line?*

A whole succession of incorrect "solutions" will follow.

1. "solution". A straight line can intersect the contour of the 12-gon only in two points, therefore this greatest number is 2.

Commentary. The author of the "solution" has forgotten that apart from convex polygons, for which the above mentioned assertion is valid, there exist also concave polygons, whose contours can be intersected by a straight line in more than two points (see, for example, fig. 7). So, his arguments are not convincing.

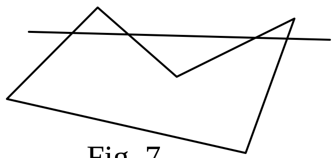


Fig. 7

2. "solution". Let us form all 4 "concaves", in this way getting 6 vertexes on one straight line. It is not possible to place more such "scallops" one after the other. Therefore the answer is 6 (fig. 8).

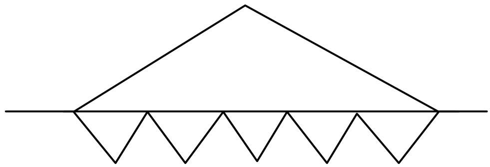


Fig. 8

Commentary. But maybe the "scallops" should not follow one another? See, for example, fig. 9, where there are 7 vertexes on one straight line.

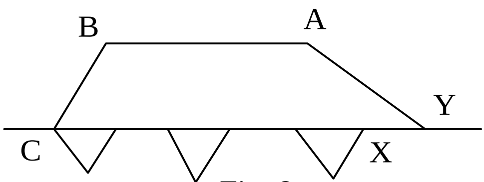


Fig. 9.

3. "solution". The maximum number is 7, as it is seen in fig. 9. When trying to join the vertex A to these 7, we fail - we get 3 consecutive vertexes X, Y, A on one straight line; it must not be like this.

Commentary. But if to the vertexes on this straight line we joined not vertex A but vertex B, as seen in fig. 10? We would get a 12-gon which has 8 vertexes on one straight line.

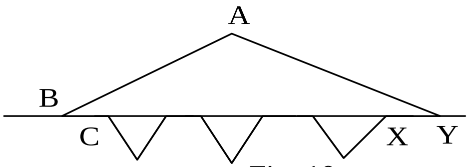


Fig. 10

Concluding commentary. How can we know, that this answer is the final? Really, it is no more possible to join vertex A as well as three lower vertexes to our straight line, but perhaps we should completely "dismantle" the whole construction and begin to form it basically otherwise? Perhaps in this way we could obtain 9 or even more vertexes on one straight line?

Out of the discussed examples we must conclude - although it seems to us, that we have obtained the best possible result, it must not be accepted without proof. Intuition may deceive us, as it just happened 3 times.

To motivate that the biggest possible value of some magnitude is n, we must motivate two things:

a) we must show an example, where this value really is n,

b) we must motivate, that in no case it (the value) can exceed n.

If even one of these parts of the proof is missing, we cannot consider the problem as solved.

Similarly, to motivate that the smallest possible value of some magnitude is n, we must motivate two things:

a) we must show an example, where this value really is n,

b) we must motivate, that in no case can it be smaller than n.

In problems of a similar type Dirichlet principle is usually used in the proofs of the second part - point b. Let us remark that such problems are more difficult than those discussed above, because of two reasons:

a) we must prove two assertions, not one,

b) the right answer is not known at the beginning, therefore, for example, using Dirichlet principle, it is not clear not only what types of "hutches" to form, but also - how many of them will be needed.

Further we shall discuss several examples.

30. example. *What largest quantity of bishops can you put on the chessboard so that they do not beat one another? (Let us presume that bishops of the same colour beat one another).*

Solution. To solve such problem, it is necessary to find a number n with two properties:

1) n bishops can be placed on the chessboard so that they do not beat each another,

2) we can prove that it is not possible to place more than n bishops so that they do not beat one another.

Let us prove that the answer of the problem is the number 14. Fig. 11 shows how we can place 14 bishops so, that they do not beat one another.

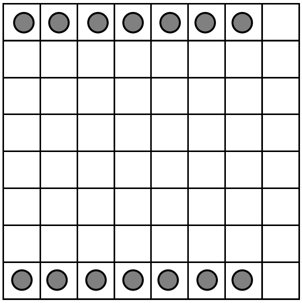


Fig. 11

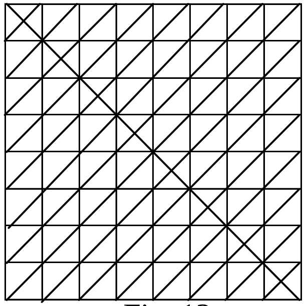


Fig. 12

To prove that 14 is the biggest possible number of bishops, we shall apply Dirichlet principle. Really, it is possible to cover the whole chessboard with "diagonals" in the way shown by fig. 12. Every such chessboard square belongs at least to one of the diagonals (some - to two diagonals). If we place more than 14 bishops on the chessboard, then, according to Dirichlet principle, at least two of them will be located on one diagonal with no other bishop between them, so they will beat each other.□

31. example. *A square consists of 8 x 8 squares. What smallest quantity of "corners" (for a sample see fig. 13) can be cut out of the square so, that it is no more possible to cut any such corner out of the remaining part of the square?*

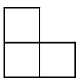


Fig. 13

Solution. We shall show that this smallest number is 11.

a) We can see in fig. 14 how to achieve the demanded result by cutting out 11 corners. The coloured squares are not cut out.

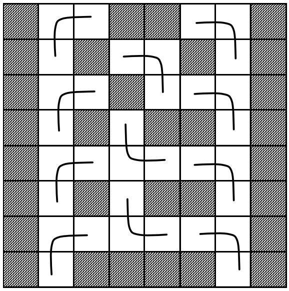


Fig. 14

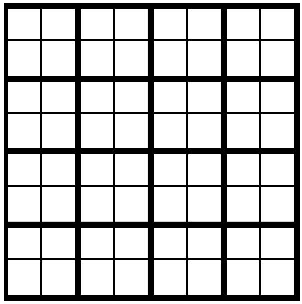


Fig. 15.

b) Let us prove that a smaller quantity of corners is not enough. If we cut out no more than 10 corners, then at least $64-10\cdot 3=34$ squares will not be cut out and remain over. They are somehow distributed into sixteen 2 x 2 square quadrates as seen in fig. 15. As $34>16\cdot 2$, then in accordance with $\mathbf{D_2}$ there would be no less than 3 uncut squares in at least one quadrate. It is clear that they form a corner.□

32. example. *What smallest quantity of squares must be painted in the quadrate consisting of 8 x 8 squares, so that in the unpainted part it would not be possible to place any rectangle consisting of 3 x 1 square?*

Solution.

a) It is enough to paint 21 square; see, for example, fig. 16.

b) It is not enough to paint less than 21 square. Let us discuss fig. 17. Each of the 21 rectangle, parameters 3 x 1, shown in this figure must have at least one painted square. Therefore, there must be at least 21 painted square altogether. (We applied Dirichlet principle: if there were ≤ 20 painted squares, then one of them should be in at least 2 rectangles shown in fig. 17. But it cannot be like that.) □

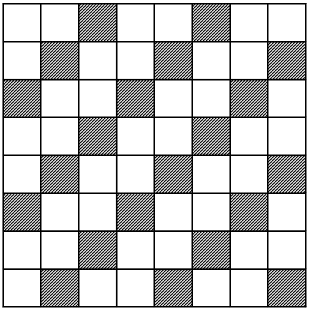


Fig. 16

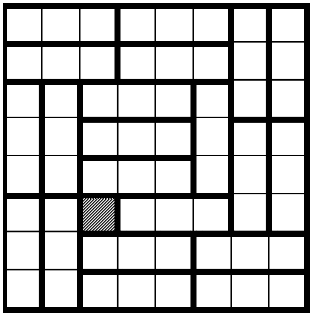


Fig. 17

33. example. *The code of each soldier in the army of Illiria is a 6-digit string (admissible are also strings beginning with one or more zeros, where digits recur etc.). Every two strings must differ in at least two positions.*

What biggest number of soldiers can be in the army of Illiria?

Solution. First, let us prove that there cannot be more than 100 000 soldiers. Let us take into consideration that there is exactly one million 6-digit strings altogether - from 000 000 to 999 999 included. Let us discuss the "tail" of each string - the string formed by the last 5 digits. So, for example, the "tail" of the string 306 977 is 06977. It is clear that exactly 100 000 "tails" are possible - from 00 000 to 99 999. If there were more than 100 000, then in accordance with $\mathbf{D_1}$ the strings of at least two soldiers would have the same “tails”. Then these strings would differ by no more than one digit - the first one, but that is not permitted.

Now we shall show how to build up 100 000 strings, which meet the condition about the difference in at least two places. Let us discuss all the possible “tails” (there are 100 000 of them) and write the last digit of the digital sum of each “tail” in front of it (the “tail”). (For example, if the “tail” is 32 726, then in front of it we write the last digit 0 of the sum 3 + 2 + 7 + 2 + 6 = 20, and obtain the string 032 726.)

It is easy to understand: if 5 positions of two 6-digit strings formed in this way coincide, then the sixth position would also coincide. But each two strings obtained in this way differ at least in one position, because they have been obtained from different “tails”. So, they coincide in no more than four positions, i.e., they differ in at least two positions, which we had to prove.□

34. example. *There are 5 different machine-tool benches in the workshop. 8 workers are hired in it. The training of one worker for operating 1 bench costs 1000 lats. It is known that every day only 5 workers will come to work, but it is not known in advance - which workers. (The absent workers need not be the same every day.) What are the lowest training costs, which can guarantee the operation of all the machine-tool benches every day, irrespective of which workers have come to work? (One worker can operate only one bench every day.)*

Solution.

a) On the table of fig. 18 we can see how to achieve the aim with costs of Ls 20 000. A cross in the square means, that the respective worker can operate the proper tool-bench. Really, if all five "particularly specialized" workers (D, E, F, G, H) come to work, then everybody gets his own bench; if some of them fail to come to work, then the same number of "Universal specialists" have come instead of them and can substitute them in an arbitrary order.

Benches \ Workers	A	B	C	D	E	F	G	H
1	x	x	x	x				
2	x	x	x		x			
3	x	x	x			x		
4	x	x	x				x	
5	x	x	x					x

Fig. 18

b) Now we shall show that Ls 19 000 (i.e., 19 trainings) is not enough. Let us presume that only 19 trainings have been carried out. As $19 < 5 \cdot 4$, hen there is a machine-tool bench, which no more than 3 workers can operate (if every bench could be operated by at least 4 workers, then the total number of trainings would be at least $5 \cdot 4 = 20$). If on some day exactly these 3 workers fail to come, the respective bench can not be used.□

<u>35. example.</u> *What biggest number of natural numbers not exceeding 100 can be chosen so, that no sum of any two chosen numbers would be equal with some third chosen number? (It is not allowed*

to build up the sum of number with the number itself.)

Solution.

a) Choosing 51 number 50; 51; 52; …; 98; 99; 100, the sum of no two chosen numbers has been chosen, because the sum of **each** two chosen numbers **exceeds** 100.

b) Let us prove that it is not possible to choose more than 51 numbers. Let us mark the biggest chosen number with k. We shall discuss two subordinate cases.

b_1) k is an even number, k = 2m. Then $m \leq 50$. Let us discuss the "hutches" $(1 \text{ and } 2m-1)$; $(2 \text{ and } 2m-2)$; $(3 \text{ and } 2m-3)$; …; $(m-1 \text{ and } m+1)$. There are m - 1 "hutches" altogether. As the sum of numbers of every "hutch" is 2m (but 2m is among the chosen numbers!), then in accordance with Dirichlet principle no more than m - 1 numbers can be chosen from the "hutches". Adding also numbers 2m and maybe m (numbers exceeding 2m are not chosen, because 2m was **the biggest** chosen number), we get that the common number of chosen numbers does not exceed $(m-1)+1+1=m+1$, i.e., it does not exceed 51.

b_2) k is an odd number, k = 2m + 1. As $2m+1 \leq 100$, then $2m \leq 99$ and $m \leq 49\frac{1}{2}$; m is a whole number, therefore from here it follows that $m \leq 49$. Discussing the "hutches" $(1 \text{ and } 2m-1)$; $(2 \text{ and } 2m-2)$; …; $(m-1 \text{ and } m+1)$, like in b_1 case we find, that no more than m + 1 numbers have been chosen; so in this case the number of chosen numbers does not exceed even 50.□

Problems for Independent Solution.

56. What largest quantity of

a) rooks (castles),
b) kings,
c) queens,
d) knights is it possible to place on the chessboard so, that no piece endangers the other? (We regard that the pieces of the same colour endanger one another.)

57***.** Solve the previous problem, if the pieces must be placed in the quadrate with parameters 9 x 9 squares.

58^{k}**.** What smallest quantity of

e) rooks (castles),
f) kings,
g) bishops must be placed on the chessboard so, that all unoccupied squares are endangered?

Note: a **beautiful** solution of a similar problem about knights and queens is not known.

59***.** What biggest number of squares is it possible to paint in a quadrate, which consists of **a)** 8 x 8, **b)** 7 x 7 squares so, that no “corner”, as seen in fig. 13, is completely painted?

60^{k}**.** What is the biggest number of cars, which, moving from left to right within the network of roads in fig. 19, can rearrange in any order? The cars may change their speed, but they must not make a reverse motion. At the beginning of movement the cars are standing in a line on the left from A one behind another; at the end of motion the cars are standing in a line to the right from B one behind another. The roads are so narrow that the leaving behind is impossible.

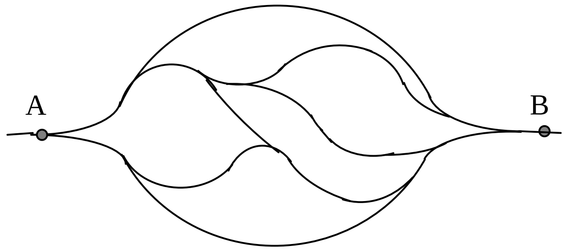


Fig. 19

61[k]**.** A town consists of 20 x 20 identical quadratic blocs; the length of the side of the bloc is 100m. The streets bound the blocs; a street leads also all round the town along its exterior contour.

What smallest quantity of currency exchange points must be opened in the town, so that going out from the house into the street at any place, you had to walk no more than along two sides of a bloc to get to the nearest point? (It is permitted to walk only along the streets.)

62[*]**.** Find what smallest quantity of squares must be painted in the quadrate, which consists of 8 x 8 squares, so that in the unpainted part it would not be possible to locate any

a) rectangle consisting of 1 x 2 squares,
b) rectangle consisting of 1 x 5 squares,
c) figure seen in Fig. 20. a),
d) figure seen in Fig. 20. b).

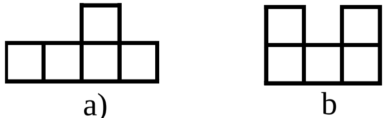


Fig. 20

63[*]**.** What biggest quantity of draughts queens is it possible to place on the black squares of the 8 x 8 draught-board so, that each queen is endangered by at least one other queen?

64*. 6 musical groups participate at the festival. Every day some of them give a performance, but the others listen. (The groups do not change their "status" within a day.) What is the smallest number of the days when it is possible to ensure, that every group has listened to all the other groups?

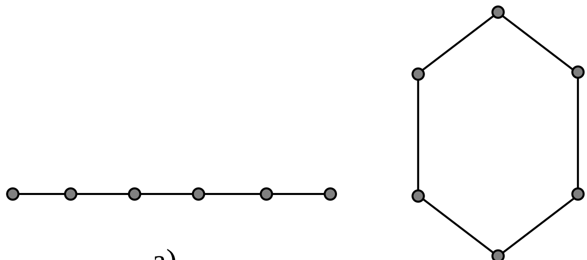
65*. There are 1995 little heaps of candies, which contain respectively $1; 2; 3; \ldots; 1994; 1995$ candies. It is possible to eat the same quantity of candies from several heaps at one stroke. What is the smallest possible number of strokes which ensures, that in all heaps there remains the same quantity of candies (maybe none)?

66. Each of the sides and diagonals of a convex octagon must be painted in the same colour, so that the segments painted alike had no common points of contact. What smallest number of colours allows to do it? (We consider that, if there exists a common point for two or more differently painted segments, it is simultaneously painted in the colours of all these segments.)

67. A quadrate consists of 9 x 9 squares. In each square there lived a little dwarf. One day all the little dwarfs decided to move to other squares, and each of them moved to such kind of square, which had a common corner with his previous dwelling (but not a common side). What is the smallest number of squares, which could remain uninhabited after this moving?

V. DIRICHLET PRINCIPLE IN THE PROBLEMS CONNECTED WITH THE ARRANGEMENTS IN STRINGS AND CYCLES

In this chapter we shall discuss the problems in which the arrangement of objects in strings (fig. 21 a)) and cycles (fig. 21 b)) is discussed. In addition, the objects are taken from two or more groups, and there exist limitations as to which groups are those, whose objects may be situated beside one another.



a)

b)

Fig. 21

36. example. *There are 2n small balls arranged in a row - white and black. No two balls of one colour can be placed beside each other. Prove that there are exactly n balls of each colour.*

Solution. We divide the positions of a row into n pairs: $(1;2)$; $(3;4)$; $(5;6)$; …; $(2n-1;2n)$. If there were more than n black balls, then in one of the n pairs there would be two black balls; they would be placed beside each other - a contradiction. If there were less than n black balls, then in one of the n pairs there would not be black balls at all; then in that pair both balls would be white and placed beside each other - a contradiction. So, there are exactly n black balls. Therefore also the number of the white balls is n.

The only arrangements of the small balls meeting the demands of the problem are seen in fig. 22 (each black ball must be followed by white, each white ball - by black).□

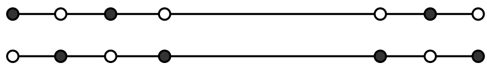


Fig. 22

37. example. *If 2n + 1 small balls - white and black - are arranged in a row, and, in addition, two balls of one colour cannot be placed beside each other, then the number of one colour balls is n, but the number of balls of other colour is n + 1. The only arrangements meeting the demands of the problem are seen in fig. 23.*

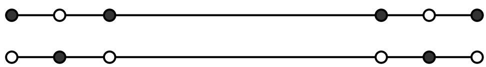


Fig. 23

Solution. We divide 2n + 1 positions into n + 1 "hutches": $(1; 2)$; $(3; 4)$; $(5; 6)$; …; $(2n-1; 2n)$; $2n+1$ (n "hutches" consists of 2 positions each, the last "hutch" - of one). Further judgement is like the solution of the 36th example.□

38. example. *If 2n small balls - white and black - are arranged in a circle, and, in addition, two balls*

of one colour are not placed beside each other, then there are exactly n balls of each colour, and they are arranged alternately.

Solution. It is like the solution of the 36^{th} example.□

39. example. *If small balls - white and black - are arranged along a circle and there are more white balls than black ones, then in some place two white balls are situated beside each other.*

Solution. Let us distribute the black balls; let us presume that their number is n. Then there are n spaces among them. We must place **more** than n white balls in these spaces; therefore a space will turn up with at least **two white** balls in it. In this space there will be the white balls beside each other. The problem is solved.□

Hence an important conclusion follows: if along a circle an odd number of white and black small balls are arranged, then in some place two one-colour balls (the colour having more balls than other) will be situated beside each other.

We recommend the reader to prove independently the results of 36. - 38. with a similar method, namely, by taking the spaces among one-colour balls as “hutches”.

We shall often use similar ideas in solving the future problems. Sometimes we shall refer to the results of examples 36. - 39., in each case specifying what we mean by balls and what - by their colours.

40. example. *Numbers from 1 to 1995 (each number is written once) are written in some order along a circle. Prove that the sum of some two numbers written beside is an even number.*

Solution. There are more odd numbers written than even numbers. Like in the solution of example 39., we prove that somewhere two odd numbers are written beside each other (the white balls - odd numbers, the black balls - even numbers). The sum of these numbers is an even number.□

41. example. *A closed broken line is drawn along the lines of a squared sheet. Prove that the number of its segments is an even number.*

Solution. No two horizontal segments are situated exactly one behind the other in this line (in such a case they would not be two segments, they would make one segment). Likewise, no two vertical segments are situated exactly behind each other. But, if the number of segments were an odd number, then in accordance with the conclusion of example 39., either one or the other of these properties would not come true; therefore the number of segments is not odd; so, it is even.□

42. example. *Can the chess knight move along all the squares of a 9x 9 quadrate so, that it gets only once into each square, and with the last move it gets back to the square where the movement began?*

Solution. We paint the squares in the usual way as a chessboard. If there existed the demanded route of the knight, all 99 squares could be arranged within the cycle in the succession the knight moves along them. As there are more squares of one colour than those of the other (because the total number is odd), then somewhere two one-colour squares would be beside each other in the cycle. But that is a contradiction, because the knight can move only from a black square to a white or from a white square to a black. So, the presumption is wrong, and there is not such a route of the knight as demanded.□

<u>43*. example.</u> *Three friends A, B, C are playing table tennis. Two of them participate in each game, but the third friend is standing and watching the game. In the next game the looser of the previous one gives his place to the friend who was watching the game he lost. It is known that A played 5 games, B - 11 games. Home many games did C play?*

<u>Solution</u>. It follows from the conditions of the problem, that at least 11 games have taken place. It also follows from the conditions, that A has not missed any two games played one after the other.

Let us look at the 6 "hutches" (they are formed of the numbers of the first 11 games): $\langle 1; 2\rangle$; $\langle 3; 4\rangle$; $\langle 5; 6\rangle$; $\langle 7; 8\rangle$; $\langle 9; 10\rangle$, 11.

In each of the first 5 "hutches" there must be **at least** one game played by A; in accordance with $\mathbf{D_3}$ in each of them there is **exactly** one game played by A. Therefore A has not taken part in the 11^{th} game. From here it follows that:

1) there has not been the 12^{th} game (otherwise A would not have participated in the games 11 and 12, played one after the other); so, there have been 11 games altogether, and B has participated in all of them,

2) out of these 11 games B has played five with A, so, he played the other six games with C; conclusion - C has played 6 games.□

<u>Commentary</u>. Let us take into consideration that we can even conclude what the results of the first 10 games will be - B has won all these games. There is

no clarity about the last, the 11^{th} game - either B or C may have won it.

44*. example. *Natural numbers from 3 to 13 are written along a circle (each number is written once). Can it happen that no two numbers written beside each other differ by no less than 3 and by no more than 5?*

Solution. The only admissible differences of the numbers written beside each other are 3; 4; 5. Let us discuss the numbers 3; 4; 5; 11; 12; 13. No two of them can be written beside each other. So, in 6 spaces among them there must be at least one other number. But for filling these six spaces we have only 5 numbers: 6; 7; 8; 9; 10. So, the demands of the problem are unrealizable.□

45. example. *Given are 55 natural numbers which do not exceed 100. Prove that among them it is possible to find such two numbers, whose difference is 9.*

Solution. Each of these 55 numbers gives some remainder, when divided by 9. The remainders can have 9 values altogether: $0; 1; 2; \ldots; 7; 8$. As $55 = 9 \cdot 6 + 1$, then in accordance with $\mathbf{D_2}$ at least one remainder will be encountered at least 7 times. We shall denote this remainder by r.

From 1 to 100 there are 11 numbers divisible by 9, 12 numbers which give the remainder 1, when divided by 9, and 11 numbers which give the remainders 2; 3; …; 7; 8 when divided by 9. (Make sure about it independently!) Let us write down in a line 11 numbers, which give the remainder r when divided by 9: $0 \cdot 9 + r$, $1 \cdot 9 + r$, $2 \cdot 9 + r$, …; $10 \cdot 9 + r$.

Those 7 of the given 55 numbers, which, when divided by 9, give the remainder r, are in this row. Judging like in the 36^{th} example we get, that two of them are beside each other in this row. Their difference is 9.□

46^k. example. *A rectangle consists of 4 x 1995 squares. Can the chess knight move along all the squares (moving only once into each square) and with the last move return to the square, where it began its movement?*

Solution. Let us take into the consideration, that painting the squares in black and white as a chess-board, which helped in example 42., does not help this time - in rectangle the number of black and white squares is the same. Therefore we shall do otherwise.

First we shall divide all the squares into two groups: interior and exterior (see fig. 24). Apparently, the number of interior and exterior squares is identical.

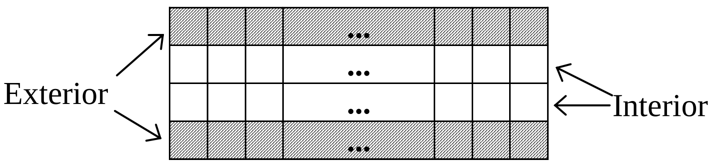


Fig. 24

Let us observe that the knight can move from the exterior square only into the interior one. So, there are at least 2 x 1994 interior squares, where the knight must move along its route leaving the 2 x 1994 exterior squares (it must move from each exterior square into another interior square). Therefore, although there exist the moves of the knight leading from an interior square to the interior one, it must not use them - then in accordance with $\mathbf{D_1}$ the knight would get more than once into some interior square, but that must not happen. For that reason all the moves of the knight from interior squares lead to the exterior squares. In accordance with the solution of the 38th example the squares along the knight's route take up their places like this:

(1) interior $\rightarrow$ exterior interior $\rightarrow$ exterior $\rightarrow \ldots \rightarrow$ interior $\rightarrow$ exterior□ (we do not show the return to the starting square; we can take it that the cyclical route begins from the interior square).

Now we shall paint the squares of the rectangle in the usual way of a chess-board. It is clear that number of black and white squares is the same. Like in the 38th example, we get the following arrangement of squares along the knight's route:

(2) black $\rightarrow$ white $\rightarrow$ black $\rightarrow$ white $\rightarrow \cdots \rightarrow$ black $\rightarrow$ white

or

white $\rightarrow$ black $\rightarrow$ white $\rightarrow \cdots \rightarrow$ black $\rightarrow$ white $\rightarrow$ black.

Comparing (1) and (2) we get, that all the exterior squares on the chess knight's route are of the same colour, but all the interior squares on the route - in the other colour. But the exterior squares are both black and white. So the knight does not move along all the squares of the rectangle. We have got a contradiction. □

<u>47^k. example.</u> *Given are 33 different natural numbers; none of them exceeds 100. Prove that among them it is possible to choose such two numbers, whose difference is 8, 9 or 17.*

<u>Solution</u>. Let us distribute all the natural numbers from 1 to 100 into four groups with 25 numbers in each:

$\{1; 2; \ldots; 25\}$, $\{26; 27; \ldots; 50\}$, $\{51; 52; \ldots; 75\}$, $\{76; 78; \ldots; 100\}$. As $33 = 4 \cdot 8 + 1$, then at least in one of these groups there are no less than 9 of the given 33 numbers; let us presume that there are n of them. We shall reduce all these numbers by one and the same magnitude so, that the smallest of them becomes 1 (in result of such operation the differences of the discussed numbers do not alter). We shall get n numbers

$$1 = x_1 < x_2 < \ldots < x_n \leq 25, \quad n \geq 9.$$

If any of the numbers x_i is 9, then the difference is found: $x_i - x_1 = 9 - 1 = 8$. If there is not the number 9 among the numbers x_i, we discuss the trios:

$$\{1; 10; 18\}$$
$$\{2; 11; 19\}$$
$$\{3; 12; 20\}$$
$$\{4; 13; 21\}$$
$$\{5; 14; 22\}$$
$$\{6; 15; 23\}$$
$$\{7; 16; 24\}$$
$$\{8; 17; 25\}$$

There are 8 of them, and they contain all x_i, $1 \leq i \leq n$. As there are at least 9 numbers x_i, then some trio contains two of them. The difference of these numbers is 8, 9 or 17, and this is what is needed.

Problems for Independent Solution.

68°. Natural numbers from 1 to 17 are written along a circle in some kind of order (each number is written once). Prove that the sum of some two numbers, written beside each other, is an even number.

69. Around the round table there are sitting 1995 little dwarfs: Pukkas, Rotivappas, Shillishallas and Snurres. It is known that Pukkas are not sitting beside Rotivappas, and Shillishallas are not sitting beside Snurres. Prove that in some place two representatives of one tribe of the dwarfs are sitting beside.

70*. 100 children are standing along a circle: 41 boys and 59 girls. Prove that it is possible to find two boys between whom exactly 19 other children are standing (no matter if they are girls or boys).

71^o. The rook made some moves and returned to the square of the start. No two moves in succession were made along one straight line. Could the rook make exactly 1995 moves?

72. Can the chess knight move along all the squares of a 7 x 7 square quadrate so, that that it got to each square exactly once and with the last move it got back to the starting square?

73*. Three friends A, B, C are playing tennis according to the rules described in example 43. It is known that they played 10, 15 and 17 games respectively. Who lost the second game?

74*. Is it possible to distribute natural numbers from 1 to 13 along a circle according to the demands of example 44 (each number is placed exactly once)?

75^o. Given are 16 natural numbers not exceeding 30. Prove that among them it is possible to find two numbers, whose difference is 5.

76^k. The king moved along all the squares of a 9 x 9 square quadrate, getting only once into each square. Not once did the king return to the square where it started the move. What was the biggest possible length of the broken line made by the king? (We take it that the length of the side of the square is 1; we measure the length of the king's move by the distance between the centres of the respective 2 squares.)

77*. A quadrate consists of n x n squares; they are painted in the way of a chess-board so that corner squares are black. By one move a chess-man can get from one square into the other, which has a common corner with the square of the chess-man's location, but not a common side. What is the smallest number of moves by which the chess-man can move along all the black squares starting its move from the black square? Solve this problem if

a) n = 8, b) n = 9, c) n = 10.

78*. A quadrate consists of 13 x 13 squares. A piece "the lion" is in one of the squares. With one move the lion can simultaneously move horizontally by m squares and vertically by n squares, where m and n are arbitrary natural numbers, $2 \le m \le 9$, $2 \le n \le 9$. (The numbers m and n can change in each move as the lion likes it. Besides, the lion may move to some square x only then, if in accordance with the above mentioned condition it might move also into the square which is symmetric with regard to one fixed diagonal of the quadrate.) Can the lion move along all the squares of the quadrate, getting to each square exactly once? It is not obligatory to return to the starting square by the last move.

79*. A quadrate consists of 12 x 12 squares. With one move a piece "the tiger" is simultaneously moving by 2 squares horizontally and by 3 squares vertically or by 2 squares vertically and by 3 squares horizontally. Can the tiger move along all the squares getting into each square exactly once and with his last move returning to the initial cell?

80*. What biggest number of natural numbers not exceeding 100 is it possible to choose so, that none of the chosen numbers exceeds the other twice?