The LAIMA series

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DIRICHLET PRINCIPLE

Part II

Theory, Examples, Problems

Experimental Training Material

Translated from Latvian by M. Kvalberga

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The book is an advanced teaching aid in mathematics for elementary school students. It contains theoretical considerations, examples and problems for independent work. It can be used as a supplementary text in classroom or for individual studies, including preparation to mathematical olympiads.

The final version was prepared by Ms. Inese Bērziņa

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About LAIMA series

In 1990 international team competition “Baltic Way” was organized for the first time. The competition gained its name from the mass action in August, 1989, when over a million of people stood hand by hand along the road Tallin - Riga - Vilnius, demonstrating their will for freedom.

Today “Baltic Way” has all the countries around the Baltic Sea (and also Iceland) as its participants. Inviting Iceland is a special case remembering that it was the first country all over the world, which officially recognized the independence of Lithuania, Latvia and Estonia in 1991.

The “Baltic Way” competition has given rise also to other mathematical activities. One of them is project LAIMA (Latvian - Icelandic Mathematics project). Its aim is to publish a series of books covering all essential topics in the area of mathematical competitions.

Mathematical olympiads today have become an important and essential part of education system. In some sense they provide high standards for teaching mathematics on advanced level. Many outstanding scientists are involved in problem composing for competitions. Therefore “olympiad curricula”, considered all over the world, is a good reflection of important mathematical ideas at elementary level.

At our opinion there are relatively few basic ideas and relatively few important topics which cover almost all what international mathematical community has recognized as worth to be included regularly in the search and promoting of young talents. This (clearly subjective) opinion is reflected in the list of teaching aids which are to be prepared within LAIMA project.

Fourteen books have been published so far in Latvian. They are also available electronically at the web - page of Latvian Education Informatization System (LIIS) http://www.liis.lv. As LAIMA is rather a process than a project there is no idea of final date; many of already published teaching aids are second and third versions and will be extended regularly.

Benedikt Johannesson, the President of Icelandic Society of mathematics, inspired LAIMA project in 1996. Being the co-author of many LAIMA publications, he was also the main sponsor of the project for many years.

This book is the second LAIMA publication in English. It was sponsored by the Scandinavian foundation “Nord Plus Neighbours”.

Foreword

This book is intended for the pupils who have an extended interest in mathematics. It can be used also by mathematics teachers and heads of mathematics circles.

For understanding the contents of the book and solving the problems, it is enough to master the course of a 9-year school. The majority of problems and examples can already be solved by 7^{th} form pupils, a great part - even by 5^{th} and 6^{th} formers.

The book contains theoretical material, examples and problems for independent solution. More difficult problems and examples are marked with asterisk **(*)**, entirely difficult - with the letter **"k"**.

We advise the readers to work actively with the book. Having read the example, please try to solve it independently, before you read the solution offered by the authors. However, you must definitely read the authors' solutions - there may turn up ideas and methods of solution, which have been unknown to you before.

The main thing that you should pay attention to, is the general line of the applied judgements, not just abstract formulations of theorems.

In each chapter, among the problems offered for independent solution, there are many problems which differ just in unessential details from the examples analysed in the text. These problems are marked with a small circle **(^{o})**.

After finding a solution or after reading the solution of the example or the problem, always think over:

“Wasn’t it possible to solve the problem otherwise?”

“Could it be possible to prove a stronger (more difficult) result with the same judgement?”

“What similar problems could it be possible to solve by judging the same way?”

We shall give short hints to some problems in the part III.

A wide range of literature has been used in writing this book, also materials of mathematics Olympiads of more than 20 countries. The sources will be mentioned in part III.

As far as we know, it is for the first time that Dirichlet Principle is discussed so widely, sistematizingly and methodically.

Introduction

There are many hundreds of methods developed in mathematics, which are successfully applied in the solution of different problems. The number of such methods is steadily growing. Usually each method is envisaged for solving a comparatively small class of problems, and it is developed in accordance with the peculiarities and specific characters of this class.

However, in mathematics there are also such methods, which are not connected with some specific group of problems; they are used in most different branches. Actually these are not just methods of mathematics, but ways of thinking which people use in solving of mathematical problems as well as in other situations of life. Getting acquainted with such methods is necessary for any intellectual person.

Many theoretical methods in mathematics (and also in life!) are based on such a principle: "**in order to accomplish great things one must concentrate big enough means in at least one direction.**" One must certainly specify the notion "great things", "direction", "big means" in every specific situation. This book shows how to do it in some mathematical situations.

VI. DIRICHLET PRINCIPLE IN THE PROBLEMS CONNECTED WITH THE CONCEPT OF GRAPH

VI.1. BASIC CONCEPTS

A graph is a picture which we get by drawing some (maybe only one) points and connecting some of them with lines (it may happen that no line is drawn). some examples of graphs are shown in Fig. 25 a), b), c).

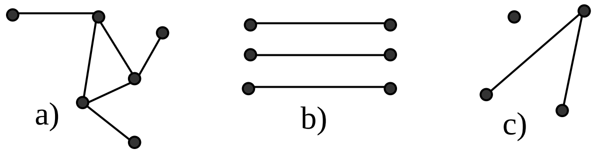


Fig. 25

The points are called **nodes**, but the lines that connect them are **edges**.

If an edge is drawn between two nodes, then it is only one of this kind.

If no special contrary notice is given, we shall not draw the edges whose both ends are on the same node.

When you look at the graph, it is not important how the nodes are placed and how the edges are drawn between them- if they are curved or straight! Only it is important, which nodes are connected by the edge and which are not. So, both graphs shown in Fig. 26 are regarded as one and the same graph.

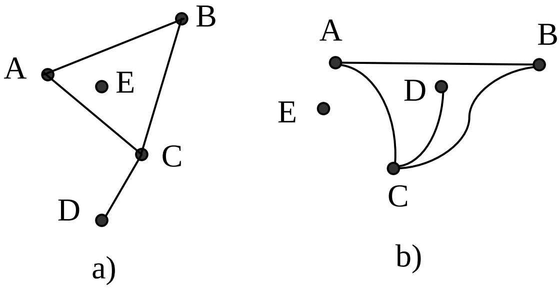


Fig. 26

The graphs are used to show the relations which can exist or not exist between the pairs of some kinds of objects. For example, if the points represent the people, we can make an agreement: we draw the edge between 2 points if the respective persons are friends, and we do not draw the edge if they are not friends. If this agreement is valid, then, for example, the graph of Fig. 27 shows that Pēteris (Peter) is a friend of Jānis (John), Kārlis (Charles) and Aivars (Ivor), but he is not a friend of Juris (George).

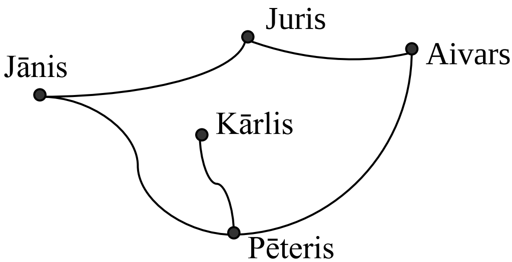


Fig. 27

Almost everybody has seen schemes of roads at bus stations, where the nodes of graphs show towns or villages, but the edges show the roads connecting them. The Fig. 28 graph shows which numbers from 1 to 7 have 1 as their greatest common measure (GCM) (they are connected by edge) and which have not.

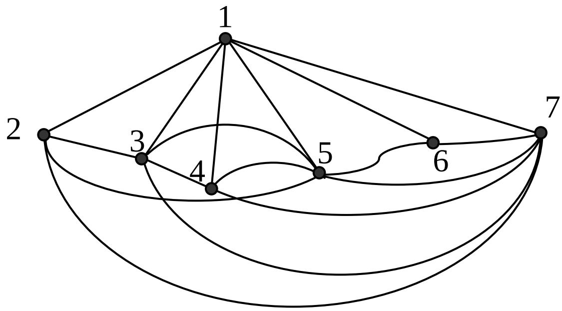


Fig. 28

Anybody of you could go on with the list of such examples for long.

From Fig. 28 we see that we are not worried about the fact that 2 edges intersect each other (for example, edges 2-5 and 3-7). In distinction to nodes, we shall not mark as black circles such common points of edges in the figure.

Sometimes we shall regard the edges as painted in two or more colours (for example, green between the nodes representing friends, and red between the nodes representing "no-friends"). As we cannot show the colours in a black-and-white text, in such situations we shall draw the edges as different types of lines (straight, undulating and the like).

The number of edges comprised by the node is called the **degree** of the node. For example, in Fig. 26 the A and B nodes' degrees are 2, the C node degree is 3, the D node degree is 1, the E node degree is 0.

The node V degree will be marked p(V).

VI.2. PROBLEMS WHERE THE CONCEPT OF NODE DEGREE OF THE GRAPH CAN BE USED

The following theorem is used in the solution of the problems of this item.

Theorem about identical degrees (TID theorem)

In every graph with at least two nodes it is possible to find two nodes with identical degrees.

▼Proof. Let us presume that there are n nodes in the graph, $n \geq 2$. Each node can be connected with $n-1$ of others at the utmost, therefore no node degree exceeds $n-1$. So, all node degrees can take only the values $0; 1; 2; \ldots; n-2; n-1$, consequently n different values.

Let us presume the opposite, that all the node degrees are different. In accordance with Dirichlet principle **D₃**, there is one node with the degree 0, one with the degree 1, …, one with the degree $n-1$. But it is not possible: if any node degree (let us mark it A) is 0, then no degree of any other node X is identical with $n-1$; really, X is not connected with A at least, therefore $p(x) \leq n-2$ (X may be connected with all the nodes except with itself and A). We have got a contradiction, so our presumption is wrong, and the degrees of all the nodes cannot be different. The theorem is proved.▲

We shall show how this theorem is used to solve the problems.

48. example. *There are 25 pupils in the class. Prove that two of these pupils have the same*

number of friends in this class (we regard that if A is B's friend, so B is also A's friend).

1. solution. Let us show the pupils as points. We draw edges between the nodes showing they denote friends. For each pupil the number of his friends is identical to the respective node degree. In accordance with the **TID** theorem, there are two nodes with identical degrees; the respective pupils have identical number of friends. The problem is solved.□

It is useful to solve this problem also directly, not referring to the **TID** theorem. Let us discuss the following solution.

2. solution. No pupil can have less than 0 friends and more than 24 friends; so, the number of friends can take 25 different values $0; 1; 2; \ldots; 23; 24$. The only possibility to avoid the situation of 2 pupils having identical number of friends is to achieve the situation where one pupil has 0 friends, one pupil - 1 friend, one - 2 friends, …, one has 23 friends, one -24 friends (we refer to Dirichlet principle $\mathbf{D_3}$). But it is not possible: if somebody has 24 friends, then he is a friend to all the others, and there is not such a pupil who has not any friends at all - everybody has at least one friend (the one who is friend to everybody). The problem is solved.□

As you see, the solution of the problem with the help of the **TID** theorem is shorter but "less concrete". We recommend that you try to solve also directly each problem which you are going to solve with the help of the **TID** theorem.

49*. example. *1000 scientists gathered at the congress. In result of inquiry it was stated: if any two scientists have a common acquaintance at the* *congress, then they have not an identical number of acquaintances. Besides, there are at least two scientists who know each other. Prove that it is possible to find the scientist who has exactly one acquaintance at the congress.*

Solution. Let us make a graph; nodes will show scientists, edges - acquaintances. Let us discuss node A whose degree is **the biggest**. From the rules of the problem it follows that $p(A) > 0$. If $p(A) = 1$, A can be taken as the scientist wanted. If $p(A) = n$, where $n > 1$, we shall discuss nodes $B_1, B_2, \ldots, B_n$, which are directly connected with A. It is clear that $p(B_1) \geq 1$, $p(B_2) \geq 1$, …, $p(B_n) \geq 1$ (all B_i are connected with A); just like $p(B_1) \leq n$, $p(B_2) \leq n$, …, $p(B_n) \leq n$, because n is the biggest node degree. Therefore $p(B_1)$, $p(B_2)$, …, $p(B_n)$ are different because $B_1, B_2, \ldots, B_n$ all are acquainted with A. Therefore, according to Dirichlet principle **D_3**, among $p(B_i)$ **all** the possible values are found including also 1. The respective scientist B_i is the wanted one. The problem is solved.□

We recommend the reader to "translate" the solution independently from the "graph language" into the ordinary language.

Problems for Independent Solution.

81°. Several delegates exchanged handshakes at the conference (each couple shook hands no more than once). Prove that it is possible to find two delegates who have had the same number of handshakes.

82. 20 teams are participating in football tournament; every team has to play a game with each other team.

Prove: no matter how the tournament is organized, at any moment you will be able to find two teams who have played the same number of games (maybe 0) by this moment.

83. 10 circles are drawn in the plane. Prove that it is possible to find two among them which contact the same quantity of the drawn circles.

84[*]**.** $(m-1)\cdot n+1$ persons have arrived at the theatre performance. Prove: among them it is possible to find either m persons, who in couples do not know one another, or find such a person who knows at least n of the other persons.

85. 25 teams participate in a tournament; every team must play one game with each other team. Prove: at any moment you can find either a team which has played at least 5 games, or 5 teams which have not played any game among themselves.

86[k]**.** In each of 3 classes there are 30 pupils. Every pupil has exactly 31 friend altogether in the two other classes (not in his class). Prove: in each class you can find one pupil so, that all three of them, taken by couples, are friendly.

VI.3. PROBLEMS TO BE SOLVED BY USING GRAPHS WITH COLOURED EDGES

In this item we shall discuss the graphs in which **every** two different nodes are connected by an edge. Each edge is painted in one of the two colours (sometimes we shall use a bigger number of colours): red or green. We shall show the red edges by wavy

(~~~~~~~~) lines, the green edges - by straight lines (__________).

50. example. *6 persons are riding in a bus. Prove that it is possible to find among them either 3 persons who all are acquainted with one another or such 3 persons among whom there are no 2 persons acquainted with each other.*

Solution. Let us show the persons by the nodes of graph. We shall draw a red edge between the nodes if the respective persons do not know each other, and a green edge if they do. This way every two points are connected by either a green or a red line. What we must prove we can express as follows: it is possible to find 3 nodes all of which, among themselves, are connected by the edges of the same colour.

We are discussing an arbitrary node A. There are 5 edges leading out of it. In accordance with Dirichlet principle $\mathbf{D_2}$, **at least** 3 of them are either in one or the other colour. Let us presume that it is possible to find 3 red edges (the other situation, when it is possible to find 3 green edges, is quite analogous).

We shall discuss the final points B, C, D of these red edges (see Fig. 29 a)).

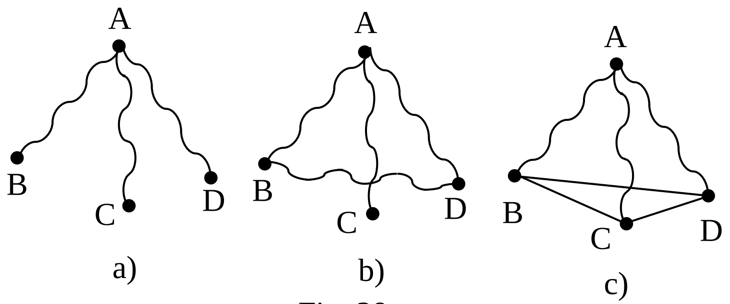


Fig. 29

If **just two** of them are connected by a red edge, then "a red triangle" is formed (for example, see Fig. 29 b)). It remains to discuss the situation when all the points B, C, D are connected among themselves by green edges. But then "a green triangle" BCD is formed (see Fig. 29 c)). The problem is solved.□

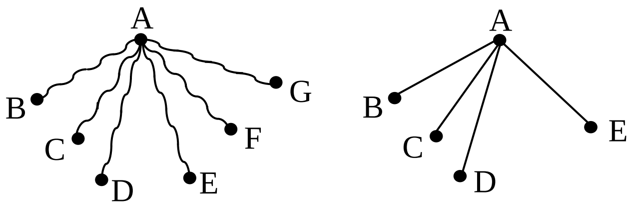
<u>51. example.</u> *10 persons are riding in a bus. Prove that it is possible to find among them 3 persons who all are acquainted with one another or 4 such persons among whom no 2 persons are acquainted with each other.*

<u>Solution</u>. Let us show the persons and their acquaintanceship just like in example 50. We must prove: in the respective graph it is possible to find either 3 nodes all connected by green edges, or 4 nodes which are all connected by red edges among themselves.

We choose an arbitrary node A. Out of it there lead 9 edges. **We assert: among them it is possible to find either 6 red or 4 green edges.** Really, if there were no more than 5 red edges and no more than 3 green edges, then the total number of edges leading out of node A would not exceed 8. But we know that there are 9 of them (or, to say it differently, $p(A) = 9$).

Let us discuss both possibilities separately.

1) Among the edges leading out of A, it is possible to find 6 red. We are discussing their final points (Fig. 30 a)).



a) Fig. 30 b)

Every two of the points B, C, D, E, F, G are connected by a red or green edge among themselves. In accordance with example 50, it is possible to find among them 3 points all of which are connected by the edges of the same colour. If it is possible to find 3 such points connected by red edges among themselves, then taking them together with point A, we get 4 points, which all are connected by red edges among themselves; that is what was necessary. If it is possible to find 3 such points connected by green edges among themselves, the necessary arrangement is obtained, too.

2) Among the edges leading out from A it is possible to find 4 green ones. We are discussing their final points (Fig. 30 b)). If just 2 of these final points are connected by a green edge, then, together with A, they form the wanted "green triangle". If, on the contrary, every two of the points B, C, D, E are connected by a red edge among themselves, then **these** 4 points meet the demands of the problem. The problem is solved.□

52[k]. example. *17 scientists correspond with one another - every scientist corresponds with all the other 16. They write about only 3 topics in their correspondences; each couple of scientists* *corresponds about only 1 topic. Prove that it is possible to find among them 3 scientists corresponding on one and the same topic among themselves.*

Solution. We shall show these scientists as vertexes of a graph. Every 2 vertexes are connected by an edge, which must be painted in one of 3 colours depending on the topic the couple of scientists are writing about. We must prove: it is possible to find 3 vertexes which are connected among themselves by the edges of the same colour.

We choose an arbitrary vertex A. Out of it there lead 16 edges; each of them is painted in one of 3 colours. In accordance with Dirichlet principle $\mathbf{D_2}$, among these edges it is possible to find 6 painted in the same colour (suppose, red). Let us discuss the final points of these 6 vertexes. Let us set apart 2 situations.

1) Just 2 of these final points are connected by a red edge. Then they, together with A, form the triangle wanted, whose all sides are red.

2) No 2 of these 6 final points are connected by a red edge. Then every 2 of them are connected by an edge painted in one of the two other colours. In accordance with the result of example 50, among these 6 final points it is possible to find such 3, all of which are connected among themselves by the edges of the same colour. The problem is solved.□

Commentary. You have already seen that the solutions of several other problems were based on the result of example 49. It is not a matter of chance: it is quite often that we have to use this result in the solution of other problems.

Problems for Independent Solution.

87°. 6 teams are participating in a tournament. Prove: at any moment it is possible either to find 3 teams all of whom have already played each with other, or to find 3 teams among which no 2 teams have yet played each with other.

88°. Prove: among any 6 towns it is possible either to find such 3 towns that, among themselves, by couples all are connected by airlines; or to find such three towns that have no direct airlines among them (we suppose that all airlines are return airlines).

89°. Prove: among any 10 segments it is possible either to find such 3 which intersect by pairs, or such 4 segments out of which no 2 intersect.

90ᵏ. Prove: among any 9 persons it is possible either to find such 3, all of whom are mutually acquainted, or such 4 persons out of whom no 2 are acquainted with each other.

91*. Using the result of problem 90, prove:

- **a)** among any 18 persons it is possible either to find such 4, who are all mutually acquainted, or such 4 persons out of whom nobody knows any other;
- **b)** among any 14 persons it is possible either to find such 3, all of whom are mutually acquainted, or such 5 persons out of whom nobody knows any other.

92ᵏ. Using the result of example 51, prove: if natural numbers from 1 to 16 including are divided in 3 groups, then it is possible to find such numbers as x, y and z, that $x + y = z$ (may be also $x = y$) in one group.

`Reference`: look at the graph whose 17 vertexes are numbered from 0 to 16, and calculate for each edge the difference of numbers of its final points subtracting the bigger number by the smaller one.

VI.4. SOME MORE CONCEPTS

Up to now we discussed the graphs whose both ends are "equal in rights". Naturally, such graphs arise in the problems where "symmetric" relations are discussed, for example, friendship: if A is friendly with B, then also B is friendly with A.

However, there exist also "non-symmetric" relations: for example, if Jānītis (Johny) loves Anniņa (Annie) it does not at all mean that also Anniņa loves Jānītis (though it may be like that). Still more obvious case: if during one round tournament team A has won over team B, then team B **definitely has not beaten team A**!

To describe such situations, you can choose the direction on the edges of the graph which is marked by a pointer. For example, if the vertexes of the graph mean teams (participants of the tournament), then we can agree to choose the direction from A to B on the edge AB to say that team A has beaten team B (see Fig. 31).

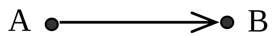


Fig. 31

Such agreements make many judgements convenient for depiction and easy for understanding. We shall use the above mentioned agreement in all the following items of VI.5.

The graph, on whose edges directions are chosen, is called **an oriented** graph (contrary to a non-oriented graph whose edges have no directions).

In the situation shown by Fig. 31 it is said that the edge leads out of the vertex A and leads into vertex B.

VI.5. PROBLEMS TO BE SOLVED BY USING ORIENTED GRAPHS

At first we shall discuss an example during whose solutions we shall turn a non-oriented graph into "something like" oriented.

53. example. *There are 10 participants in the circle of mathematics. On holidays each of them sent 5 greeting cards to the other participants of the circle. Prove that such 2 participants will turn up who have sent greeting cards to each other.*

Solution. Let us show the participants of the circle as vertexes of the graph. We shall connect every two vertexes by an edge. From each vertex 9 edges lead out, so, in total $10 \cdot 9 = 90$ points of the edges. Each edge has 2 points; therefore, there are $\frac{90}{2} = 45$ edges in total. If participant A has sent his greetings to participant B, then on the edge AB we shall draw the pointer from A to B. Each participant has sent 5 greetings, so in total we must draw $10 \cdot 5 = 50$ pointers. There are more pointers than edges, therefore in accordance with $\mathbf{D_1}$, an edge will appear where 2 pointers have been drawn. Participants, corresponding to the points of the edge, have sent their greetings to each other. The problem is solved.□

Commentary. Formally speaking, we cannot say that we have got an oriented graph as in an oriented graph **only** one direction is drawn on **each** edge.

Very extensively oriented graphs are used to describe different tournaments (in sport where drawn games are impossible) and to solve the problems about them.

In all the following examples and problems of item VI.5, where we talk about tournaments, our opinion is that each team plays with each other team exactly one game: besides, there is not a drawn game.

54. example. *8 teams participate in a tournament. Prove that, when it is over, it is possible to find such 4 teams A, B, C, D that simultaneously* $A \to B$, $A \to C$, $A \to D$, $B \to C$, $B \to D$ *and* $C \to D$ *(i.e., in the chain of teams ABCD every team has beaten all the following teams).*

Solution. We shall show the tournament as an orientated graph. There are 28 edges altogether. (We get the result $8 \cdot \frac{7}{2} = 28$ just like the number of edges in the solution of example 53). Each of these edges leads out of one of the 8 vertexes. As $28 = 8 \cdot 3 + 4$, i.e., $28 > 8 \cdot 3$, then, in accordance with $\mathbf{D_2}$, a vertex will be found where at least 4 edges lead out. We choose this vertex as A (Fig. 32).

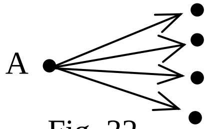


Fig. 32

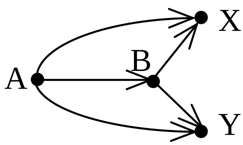


Fig. 33

We look at the final points of 4 edges leading out of A. Now we shall take interest in edges that connect these final points. There are $4 \cdot \frac{3}{2} = 6$ such edges altogether. Each of them leads out of one of 4 vertexes. As 6 > 4, then, in accordance with $\mathbf{D_1}$, a vertex will be found out of which at least 2 edges lead out within the system of the 4 final points mentioned before. We choose this vertex as B (Fig. 33) and take our interest in the final points of both given edges (Fig. 33, X and Y).

There is also an edge between X and Y. If it goes from X to Y, we choose X as C and Y as D; if it goes from Y to X, we choose Y as C and X as D (Fig. 34). The necessary result is gained, the problem is solved. □

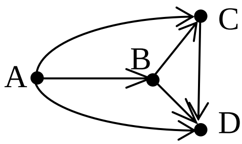


Fig. 34

55. example. *6 teams participate in the tournament. Prove that, when it is over, it is possible to find such 2 teams, that each of the remaining 4 has lost a game to at least one of these 2 teams.*

Solution. In total $6 \cdot \frac{5}{2} = 15$ games have been played in the tournament, so there are 15 wins. In accordance with Dirichlet principle $\mathbf{D_2}$, it is possible to find a team who has gained **at least** 3 victories. Further we distinguish 3 cases:

a) In reality it is possible to find the team, who has gained 5 victories, i.e., it has won **all** the games. We choose this team as one of the wanted; we choose the other one arbitrarily.

b) In reality it is possible to find the team, who has won 4 games but lost 1 game. We choose this team and the one to whom this team has lost.

c) There is team A, who has exactly 3 wins (and consequently 2 losses - let us presume it is against teams B and C). One of the teams B and C has won their mutual game; we presume that $B \to C$. Then we choose teams A and B. We have discussed all the situations, the problem is solved.□

Problems for Independent Solution.

93°. Each of 2n chatterboxes has got to know thrilling news about n others. Prove: it is possible to find 2 chatterboxes who have got to know thrilling news about each other.

94°. 16 teams participate in a tournament. Prove that at the end of the tournament you can choose 5 teams out of them and mark them A, B, C, D, E so, that in the string ABCDE each team has gained victory over all following ones.

95. 14 teams participate in a tournament. Prove that after it is over you can choose 3 teams so, that each of the other 11 has lost the game to at least one of these 3 teams.

96*. Solve the above problem, if from 30 teams you must choose such 4 teams so, that each of 26 others would have lost game to at least on of these 4 teams.

97ᵏ. Solve problem 96 if number 14 is substituted by 18, and 11 - by 15.

98*. 17 teams participate in a tournament. Prove that at any moment it is possible either to find 5 teams among whom no 2 teams have played a game with each other, or to find 5 teams which can be marked by A_1, A_2, A_3, A_4, A_5 so, that $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow A_4 \rightarrow A_5$.

VII. METHOD OF THE SUM ESTIMATION

VII.1. MORE DETAILED ANALYSIS OF DIRICHLET PRINCIPLE

(You may skip this item, when you read it for the first time!)

Suppose we must solve the following task.

P_1 *11 little bags of sugar, each weighing 1 kg, are somehow placed in 10 boxes. Prove that that there is a box containing at least 2 kgs of sugar.*

Let us give the solution.

1. step. We assert that at least in one box there is more than 1 kg sugar. Really, if in each box there were no more than 1 kg of sugar, then in all 10 boxes there would be no more than 10 kgs of sugar in total. But there are 11 kgs altogether. So, in at least one box the amount of sugar exceeds 1 kg.

2. step. If the amount of sugar in the box exceeds 1 kg, then **there is more than 1 bag in this box. So, there are at least 2** bags, so at least 2 kgs of sugar.

The problem is solved.□

Now let us discuss the following task.

P_2 *11 kgs of granulated sugar are somehow poured in 10 boxes. Can you assert that at least in 1 box there are not less than 2 kgs of sugar?*

Certainly not! It may occur that in each box there is, for example, $1\frac{1}{10}$ kg of sugar. Then in total there are 11 kgs, but in no box the amount of sugar is either 2 kgs or more than 2 kgs. Which part of our proof was valid for problem $\mathbf{P_1}$ but is not valid for problem $\mathbf{P_2}$ any more?

The first step of the solution remains correct also in this case; make yourself sure about it independently. But the second step for problem $\mathbf{P_2}$ is not useful any more. An essential difference: in problem $\mathbf{P_1}$ the amount of sugar in each box is expressed by a whole number of kilograms, but not so in problem $\mathbf{P_2}$. A whole number exceeding 1 is at least 2; on the contrary, an arbitrary number (not definitely whole) exceeding 1 may also be smaller than 2 (for example, number 1.1).

It is not difficult to understand that in reality the application of Dirichlet principle in the form of $\mathbf{D_1}$ or $\mathbf{D_2}$ in all situations is carried out according to the following scheme:

1) Some of the theorems is applied:

T1: if $x_1 + x_2 + \ldots + x_N \geq a_1 + a_2 + \ldots + a_N$, then at least for one i the inequality $x_i \geq a_i$ holds;

T2: if $x_1 + x_2 + \ldots + x_N > a_1 + a_2 + \ldots + a_N$ then at least for one i the inequality $x_i > a_i$ holds;

T3: if $x_1 + x_2 + \ldots + x_N \leq a_1 + a_2 + \ldots + a_N$, then at least for one i the inequality $x_i \leq a_i$ holds;

T4: if $x_1 + x_2 + \ldots + x_N < a_1 + a_2 + \ldots + a_N$, then at least for one i the inequality $x_i < a_i$ holds;

(These theorems are proved very simply with the help of summing up assuming the contrary; we advise the reader who wants to practise algebra to do it independently.)

2) If, in addition, it follows from the conditions of the problem that x_i are whole numbers, then out of the obtained inequalities more powerful ones are gained.

For example, from $x \geq 4.8$ in this case follows $x \geq 5$; from $x > 6$ follows $x \geq 7$; from $x < 1.2$ follows $x \leq 1$; from $x < 4$ follows $x \leq 3$ etc.

It is quite often that in solving the problems the first part of the above mentioned argument is enough; it is not necessary to approximate the number.

We shall show the examples of such solutions in the next item. The solutions resemble the application or proof of Dirichlet principle, but, to say it more simply, “the approximation is missing” or no special “hutches” are introduced like in case of Dirichlet principle; they speak about the sum itself, not about its division into “hutches”.

VII.2. EXAMPLES OF ESTIMATION OF THE SUM

56. example. *Prove: out of 10 different natural numbers it is possible to find two whose sum is at least 19.*

Solution. Let us arrange these 10 numbers in an increasing order: A < B < C < D < E < F < G < H < I < J. As A is a natural number, $A \geq 1$. As B > A, we get $B \geq 2$. As C > B, we get $C \geq 3$. In a similar way we get $D \geq 4$, $E \geq 5$, $F \geq 6$, $G \geq 7$, $H \geq 8$, $I \geq 9$, $J \geq 10$. Therefore

$I + J \geq 9 + 10 = 19$. So, we can take both bigger numbers as the wanted ones.□

<u>57. example.</u> *The sum of five numbers is 10. Prove that out of them it is possible to find two, whose sum is at least 4.*

<u>Solution</u>. We arrange the numbers in an undimishing order: $A \leq B \leq C \leq D \leq E$. We presume that $D + E < 4$. Then from these inequalities it follows that also $B + C < 4$. As $B + C < 4$ and $B \leq C$, then $B < 2$. (Really, if it were $B \geq 2$, then from $B \leq C$ it would follow $C \geq 2$ and $B + C \geq 4$.) As $A \leq B$ and $B < 2$, then $A < 2$. Summing up the inequalities $A < 2$, $B + C < 4$, $D + E < 4$, we get $A + B + C + D + E < 10$, which contradicts the given. so ,the presumption is wrong, and $D + E \geq 4$. The problem is solved.□

<u>58. example.</u> *21 boy has 2 lats* altogether. Prove that it is possible to find such two boys who have the same amount of money.*

<u>Solution</u>. Let us presume from the contrary that all the boys have different amount of money. It is clear that for each boy it can be expressed by a whole number of santims*. Judging like in the solution of example 56, we get that the common amount of money is at least $0 + 1 + 2 + \ldots + 19 + 20 = 210$ santims. It is a contradiction because the common amount of money is 200 santims. So, our presumption is wrong. The problem is solved.□

<u>59*. example.</u> *A table consists of 5 x 5 squares. Some number is written in each square; the sum S of all the written numbers is positive. Prove that the lines of the table can be rearranged in such a*

* 1 lat = 100 santims (currency of Latvia)

succession (the order of the numbers is not changed within any line) that the sum of numbers written on the main diagonal of the obtained table (see Fig. 35) is positive.

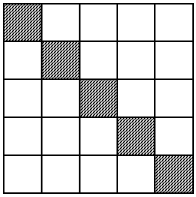


Fig. 35

1	5	4	3	2
2	1	5	4	3
3	2	1	5	4
4	3	2	1	5
5	4	3	2	1

Fig. 36

Solution. We discuss 5 different arrangements of the lines: 1) the initial, 2) the one obtained from the initial removing the upper line and placing it at the bottom, 3) the one obtained from the second removing the upper line and placing it at the bottom, 4) the one, similarly obtained from the third, 5) the one which is similarly obtained from the fourth.

We shall mark the sums of numbers of the main diagonal in these arrangements by A; B; C; D; E. It is easy to understand that each number is on the main diagonal exactly in one arrangement (in Fig. 36 in each square there is written the number of the arrangement in which it goes to the main diagonal). Therefore, $A+B+C+D+E=S$. As $S>0$, someone of the items has to be positive. The arrangement corresponding to it is good as the wanted one. The problem is solved.□

60. example. *Given is that a and b are real numbers. Prove that there exists a real root for at least one of the equations* $x^2+2ax+b=0$ *;* $ax^2+2bx+1=0$*;* $bx^2+2x+a=0$.

Solution. If even one of the numbers a, b is 0, then such is the third equation (test it independently). If $a \neq 0$ and $b \neq 0$, then all the equations are quadratic equations; their discriminants are $D_1 = a^2 - b$, $D_2 = b^2 - a$ and $D_3 = 1 - ab$. The sum of all discriminants $D_1 + D_2 + D_3 = a^2 + b^2 + 1 - a - b - ab =$ $= \frac{1}{2}\left[(a-b)^2 + (a-1)^2 + (b-1)^2\right] \geq 0$, so someone of the discriminants is not negative; the corresponding equation has real roots.□

61. example. *The table consists of 8 x 8 squares. Different natural numbers from 1 to 64 are written in them - one number in each square. Prove that there exist 2 squares with a common side, such that the difference of the numbers written in them is at least 5.*

Solution. In some square there is written 1, in some square - 64. Let us discuss the shortest “horizontal - vertical” way from the first square to the second (see, for example, Fig. 37).

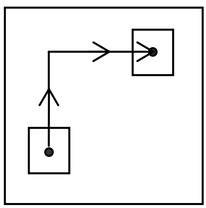


Fig. 37

This way consists of no more than 14 steps, in each step moving from the square to such a square which has a common side with the previous one. If we made each step among the squares, where the difference of the written numbers does not exceed 4, then the difference between the numbers written in the initial and in the final squares would be no more than $4 \cdot 14 = 56$; but it is $64 - 1 = 63$.

Conclusion: on the above discussed way from 1 to 64 in some step we have found two squares beside each other, where the difference of the written numbers is at least 5.□

62^k. example. *The table consists of 17 x 17 squares. In each square there is written one natural number from 1 to 17; each such number is written in exactly 17 squares. Prove that it is possible to find such a line or such a column, where at least 5 different numbers are written.*

Solution. Let us call the total number of the columns and lines, each containing at least one copy of the number n, a distribution of n, denoting it by I_n. First we shall show that the distribution of each number is at least 9. Really, if some number is found in x lines and y columns, then it is not found anywhere outside the intersection points of these lines and columns - so, it is written in no more than $x \cdot y$ squares. Therefore there must be $x \cdot y \geq 17$. From the inequality $x + y \geq 2\sqrt{xy}$ it follows that $I = x + y \geq \; \geq 2\sqrt{17} > 8$. Then $I \geq 9$, as I is a natural number. Now we shall discuss the sum $S = I_1 + I_2 + \ldots + I_{17}$. From the previously proved it follows $S \geq 9 \cdot 17 = 153$.

It is clear that $I_1 + I_2 + \ldots + I_{17} = (r_1 + r_2 + \ldots + r_{17}) + (k_1 + k_2 + \ldots + k_{17})$, where r_i is the number of different numbers in the i^{th} line and k_i is the number of different numbers in the i^{th} column (belonging of each number to some line or some column gives "investment" 1 in both the left and the right side of this equivalence). As the sum of 34 items is not smaller than 153, then at least one of these items is not smaller than $\frac{153}{34} = 4.5$. As all the items r_i and k_i are natural numbers, then the corresponding line or column is the wanted one. The problem is solved. □

63^k. example. *After the election of the parliament the deputies formed 12 factions (each deputy became member of exactly 1 faction). After the first plenary meeting the deputies' opinions changed, and they united in 16 new factions (still each deputy was member of exactly 1 faction). Prove that now at least 5 deputies are in smaller factions than immediately after the election of the parliament.*

Solution. If the deputy is member of the faction with x members altogether, we can say his importance is $\frac{1}{x}$. Let us presume that the importance of the deputies in the first distribution were $a_1, a_2, \ldots, a_N$, but in the second - $b_1, b_2, \ldots, b_N$. We must prove that for at least 5 different indexes i the inequality $b_i > a_i$ holds. It is clear that the sum of all importances in each faction is 1; therefore, in the first distribution the sum of importances of all the parliamentary deputies is 12, but in the second it is 16. The importance of each deputy in each distribution is a positive number not exceeding 1; therefore the importance of each deputy, when the distribution changes, is altering by the magnitude which is smaller than 1. If the importances of only 4 deputies have increased, then the sum of importances has increased by less than 4; so, it cannot increase from 12 to 16. Consequently the importances of at least 5 deputies have increased. The problem is solved.□

Sometimes, applying a similar method, the product is discussed instead of the sum.

64[k]. example. *It is given that 0 < a < 1, 0 < b < 1, 0 < c < 1. Prove that at least one of the numbers* $a(1-b)$, $b(1-c)$, $c(1-a)$ *does not exceed* $\frac{1}{4}$.

Commentary. The previous method of solution is not useful: we cannot prove that the sum of all numbers does not exceed $\frac{3}{4}$, because, for example, if $a = 0.98$; $b = c = 0.01$, then already the first item alone exceeds $\frac{3}{4}$. We must look for another way.

Solution. The product of all 3 investigated numbers $R = a(1-b)\cdot b(1-c)\cdot c(1-a) = a(1-a)\cdot b(1-b)\cdot c(1-c)$.
It is easy to understand that for each x there exists an inequality $x(1-x) = x - x^2 = \frac{1}{4} - \left(\frac{1}{2} - x\right)^2 \le \frac{1}{4}$. Therefore $R \le \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}$. Now it is clear that simultaneously there cannot be $a(1-b) > \frac{1}{4}$, $b(1-c) > \frac{1}{4}$; if it were like that, then it would be $R > \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}$. The problem is solved.□

Problems for Independent Solution.

99°. Prove that among 100 different natural numbers it is possible to find 3, whose sum exceeds 296.

100°. 15 boys have 100 mathematics books altogether. Prove: there are 2 boys who have the same number of books.

101°. Solve example 59, if instead of the table 5 x 5 we are discussing the table of n x n squares.

102. 7 girls have 110 dolls altogether. Prove that it is possible to find 4 girls who have at least 70 dolls altogether. It is known that all the girls have different number of dolls.

103° Given is that a, b, c are real numbers. Prove that at least one of the equations $ax^2 + 2bx + c = 0$; $bx^2 + 2cx + a = 0$, $cx^2 + 2ax + b = 0$ has a real root.

104. The table consists of 10 x 10 squares. Natural numbers from 1 to 10 are written in them (1 number in each square). Prove that there exist

a)° 2 squares with a common side and the difference of numbers written in them at least 6,

b) 2 squares with a common side or a common corner and the difference of numbers written in them at least 11,

c)ᵏ 2 squares with a common side, where the difference of numbers written in them is at least 10.

105. The table consists of 10 x 10 squares. A natural number is written in each of them. The numbers written in every 2 squares with a common side do not differ more than by 1.

a)[o] Prove that there is a number which is written in at least 6 squares.

b)[k] Prove that there is a number which is written in at least 10 squares.

106. A regular triangle ABC is divided into 25 small regular triangles as shown in Fig. 38. In each of the small triangles there is written a natural number from 1 to 25 (different numbers in different triangles). Prove that it is possible to find 2 triangles with a common side where the written numbers differ:

a) at least by 3,

b)[*] at least by 4..

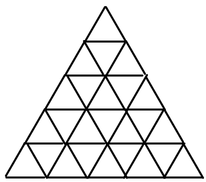


Fig. 38

107[*]**.** Given is 21 natural number; they all are different, and none of them exceeds 71. Prove that among the differences of these numbers it is possible to choose 4 differences, all of which are equal to one another.

108. Given are 20 natural numbers; they all are different, and all are smaller than 70. Prove that out of differences of these numbers it is possible to choose 4 that all are equal among themselves.

109[o]**.** At the boxing contest the boxers are divided into 12 groups of weight and they represent 6 teams. Prove: it is possible to find at least 7 such boxers who have more team-mates in this championship than contestants in their class of weight.

110^k. 512 wrestlers participate in a tournament. First they divide themselves into pairs and wrestle with each other; 256 winners wrestle in pairs again, 128 winners of these pairs wrestle again etc. (The system arranging such tournaments is called Olympic). In final there wrestle representatives of both previous pairs - winners of the semi-final. Before the beginning of the tournament each wrestler was given a qualification number from 1 to 512 (a different number to each sportsman); this number does not alter during the tournament. The contest is called interesting if the wrestlers, whose numbers do not differ by more than 30, participate in it. Prove that in the tournament there is at least 1 uninteresting contest.

111. Given is that a, b, c, d, e, f, g, h, i are some numbers. Can all the numbers aei, bfg, dhc, -ceg, -bdi, -afh be positive simultaneously?

112. Is the following pre-election promise of some party (let us keep silent - which one) decent:
"Our party proposes to achieve the objective that any working person gets more than the average monthly salary"?

VIII. DIRICHLET PRINCIPLE IN GEOMETRY

In this chapter we shall discuss the problems where Dirichlet principle or the method discussed in the previous chapter are used in solving the problems of geometry. In all examples the applications of Dirichlet principle or estimations of sums are used together with some geometrical idea or fact; we shall try to indicate it clearly in any case.

VIII.1. MAXIMAL DISTANCE BETWEEN THE POINTS OF A POLYGON

65. example. *In a 3×4 rectangle 7 points are placed (inside or on a contour). Prove that at least 2 of them are located no farther than $\sqrt{5}$ from each other.*

Solution. Let us look at Fig. 39.

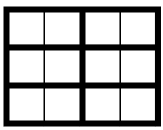


Fig. 39

In at least one of 6 rectangles there are 2 or more of the discussed points (Dirichlet principle!). As the ends of the diagonal are the farthest points in the rectangle (geometrical idea) and the length of the diagonal is $\sqrt{5}$

(Pythagorean theorem), then both these points are the wanted ones.□

66*. example. *In a 3 x 4 rectangle 6 points are placed (inside or on a contour). Prove that at least 2 of them are located no farther than $\sqrt{5}$ from each other.*

Commentary. It is not possible to use the division of the rectangle into 6 “hutches” as shown in Fig. 39 - perhaps only 1 point gets into each “hutch”.

Solution. Let us see Fig. 40 where the rectangle is divided into 5 parts. You can find at least 2 points in one part. As within each part the distance between the points does not exceed $\sqrt{5}$ (check yourself that no side and no diagonal is longer than $\sqrt{5}$), then we can take both these points as wanted ones.□

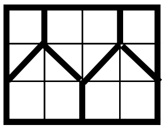


Fig. 40

Commentary. The following intuitively evident fact was used without proof in the solution of both previous examples: the longest distance between the points of a polygon is the distance between some two of its vertexes. We advise the reader to try to prove it independently.

67[k]. example. *A regular triangle T with the length of the side 1 is completely overlapped with 5 identical patterns; each pattern has the form of a regular triangle (the length of its side is not known). Probably the patterns overlap each other partially.*

Prove that it is possible to overlap the regular triangle T with 4 patterns of the same size.

Solution. Let us discuss the midpoints of the sides and the vertices of triangle T (Fig. 41[a]). Every two of these 6 points are at the distance $\geq \frac{1}{2}$ from each other. As they are covered by 5 patterns, then at least 2 points are covered by 1 pattern. The maximum distance between the points of a regular triangle (pattern) is the length of its side; we conclude that the length of the pattern's side is at least $\frac{1}{2}$ because it overlaps 2 points of the 6 mentioned above.

Triangle T (Fig. 41[b]) is divided into 4 regular triangles, and each of them can be covered by a pattern whose side is $\geq \frac{1}{2}$ long.□

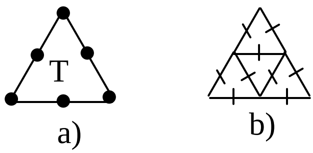


Fig. 41

VIII.2. THE APPLICATION OF PARTICULAR, SPECIALLY FORMED SYSTEM OF POINTS

The essence of the method is well enough in the heading of this chapter. We shall discuss it within examples.

68. example. *Each point of the plane is painted white, black or red (the painting is entirely arbitrary). Prove that it is possible to find 2 points which are painted in the same colour and they are at the distance of 1m from each other.*

Solution. We are discussing an equilateral triangle whose side is 1m long. It has 3 apexes; among them there are 2 painted in the same colour (Dirichlet principle $\mathbf{D_1}$). These apexes are good as the wanted points.□

69*. example. *Each point of the plane is painted white, black or red (the painting is entirely arbitrary). Prove that it is possible to find 2 points which are painted in the same colour and the distance between them is 1m.*

Solution. Let us look at Fig. 42; every 2 points connected by a segment are at the distance of 1m from another. You can form such system in the following way (see Fig. 43): at first we make 2 rhombi ABCG and AFDE, and each of them consists of 2 equilateral triangles whose each side is 1m long. After that we are turning them towards each other round point A as long as CD becomes 1m long, too. Now we are analyzing the system of 7 points shown in Fig. 42.

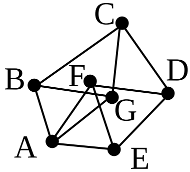


Fig. 42

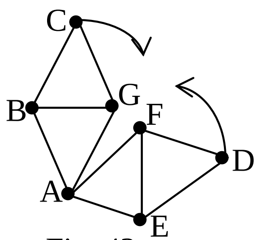


Fig. 43

Let us presume that these 7 points are painted so, that among them there are not 2 points at the distance of 1m from one another. As 3 colours are used and $7 > 3 \cdot 2$, then among these 7 points 3 will turn up painted in the same colour. If one of them is B, F, G or E, then the other two must be apexes of 1 equilateral triangle with a 1m long side (for example, if one point is B, then the other two must be EFD apexes); then among them there are two at the distance of 1m in accordance with example 68. If none of them is B, F, G or E, then they are A, C, D; but CD = 1m. We get a contradiction in all cases. The problem is solved.□

In the solving of the next example we shall use the property of vertexes of a regular pentagon: every three vertexes of a regular pentagon form an isosceles triangle.

Actually this property exists even if the centre of the pentagon is added to the 5 vertexes (see Fig. 44). Motivate it independently (it is easy to do it).

70. example. *9 vertexes of a regular 20-gon are painted black. Prove that it is possible to find an isosceles triangle whose all vertexes are black.*

Solution. Let us number the vertexes in turn with 1; 2; 3; …; 19; 20 and divide them into 4 groups $1, 5, 9, 13, 17;$ $2, 6, 10, 14, 18;$ $3, 7, 11, 15, 19;$ $4, 8, 12, 16, 20$. It is not difficult to verify that the vertexes of each group form a regular pentagon.

As $9 > 4 \cdot 2$, then 3 black points will turn up that belong to the same group. In accordance with the above mentioned property of the regular pentagon they may be accepted as the wanted.□

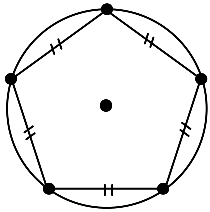


Fig. 44

VIII.3. THE PROPERTY OF VERTEXES AND SEGMENTS OF A BROKEN LINE

Let us make use of the following properties (here they follow from the definition of a broken line): no 3 vertexes of a broken line, taken in succession, are situated on one straight line; no two segments of a broken line, taken in succession, are situated on one straight line.

71. example. *What biggest number of vertexes of a 12-gon can be situated on one straight line?*

Solution. We already saw (see Part 1, p. 56) that 8 vertexes can be situated on one straight line. Let us show that 9 vertexes cannot be situated on one straight line. We shall divide into 4 groups the vertexes of a 12-gon, which are numbered in succession with numbers from 1 to 12:

1 2 3   4 5 6   7 8 9   10 11 12

Let us presume that 9 vertexes are situated on one straight line. As $9 > 4 \cdot 2$ then some 3 of these 9 vertexes belong to one group. But in that case three vertexes in succession would be on one straight line - a contradiction. So, our presumption is wrong.□

72. example. *36 points are placed in a square-like grid (see Fig. 45). What is the smallest number of segments for a closed broken line that goes*

through all the points? All the segments must be horizontal or vertical.

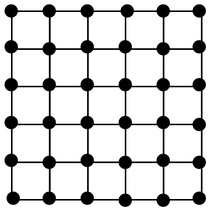


Fig. 45

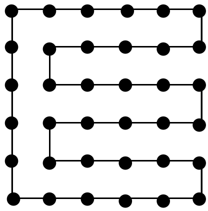


Fig. 46

Solution. The line shown in Fig. 46 has 12 segments. Let us show that a smaller number of segments is not enough. As in a closed broken line vertical and horizontal segments change by turns, so the number of segments is an even number.

Let us presume that in this line there are less than 12 links. Then there are no more than 5 horizontal and no more than 5 vertical links. As there are 6 horizontals in the grid, then there is such a horizontal H which has not a single horizontal link; as there are 6 verticals in the grid, then there is such a vertical V which has not a single vertical link. Our broken line does not lead through the point where H and V intersect. It is a contradiction. So it needs at least 12 links.

73. example. *6 straight lines are intersecting. What is the biggest possible number of sides of the polygon whose all sides are situated on these lines?*

Solution. Each straight line can intersect no more than 5 others, so no more than 5 points of intersection arise on it. They divide the straight line in no more than 6 parts. Two of them are rays. So, on each straight line there arise no more than 4 segments without common inner points. It is easy to understand that from these segments no more than 2 sides of a polygon can form, because two sides situated on one straight line cannot have common points. So ,on each straight line there are no more than 2 sides of a polygon, therefore the total number of the sides does not exceed $6 \cdot 2 = 12$. Fig. 47 shows that it is possible for such a polygon to have 12 sides. Consequently, the answer of the problem is 12.□

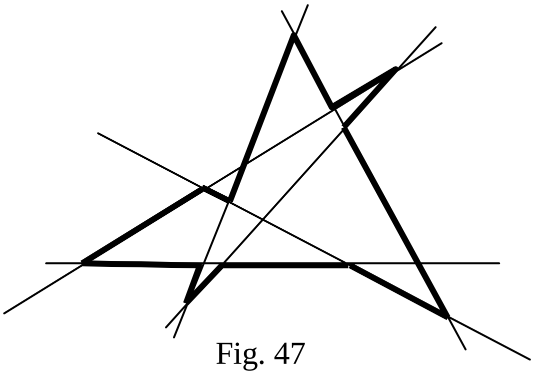


Fig. 47

VIII.4. ESTIMATION METHODS OF SUMS

The solutions of the problems of this chapter essentially use the following analogues of Dirichlet principle (we shall call them the theorems $\mathbf{D_4}$ and $\mathbf{D_5}$).

Theorem D_4

If the sum of n items exceeds S, then at least one of the items exceeds $\frac{S}{n}$***; if the sum of n items is not***

smaller than S, then at least one of the items is not smaller than $\frac{S}{n}$***.***

Theorem D_5

If the sum of n items is smaller than S, then at least one of the items is smaller than $\frac{S}{n}$***; if the sum of n items does not exceed S, then at least one of the items does not exceed*** $\frac{S}{n}$***.***

Let us show, for example, how the first part of $\mathbf{D_4}$ is proved.

Let us presume the contrary - none of n items exceeds $\frac{S}{n}$. In this case the sum of these n items does not exceed $\frac{S}{n}+\frac{S}{n}+\ldots+\frac{S}{n}$ (n times) or it does not exceed S. It is a contradiction to what is given in the terms of the theorem. So, our presumption is wrong, and there must be the item exceeding $\frac{S}{n}$. The reader himself can prove also the other theorems from the contrary with the help of addition of inequalities (it is very easy).

In specific problems S and the corresponding items can be of most different character: areas, lengths, distances, magnitudes of angles, angular magnitudes of arcs etc. With the help of theorems $\mathbf{D_4}$ and $\mathbf{D_5}$ we shall try to draw some conclusions about the area, length, angular magnitude etc. of a **particular** figure, using the information about the sum of **several** areas, lengths, angles etc.

VIII.4.1. DIRECT ESTIMATIONS OF SUMS

First, we shall discuss the problems connected with the conception of area. Here the following property is important: if the figure is divided into several parts then the area of the figure is equal to the sum of areas of separate parts.

74. example. *A square of dimensions 6 x 6 cells is cut into 9 rectangles: the cuts run only along the lines of the cells.*

Prove that there are 2 equal rectangles among the 9 cut ones.

Solution. Let us presume that all the rectangles obtained by cutting are different. The table below shows rectangles with the smallest possible areas (the unity of measure of the area is 1 cell).

Area	1	2	3	4	5	6	7
Dimensions	1 x 1	1 x 2	1 x 3	1 x 4 2 x 2	1 x 5	1 x 6 2 x 3	1 x 7

It is evident: even if we choose 9 different rectangles with the smallest possible areas, the sum of their areas is $1+2+3+4+4+5+6+6+7=38>36$. So it is not possible simultaneously to obtain 9 different rectangles from the given square. Therefore 2 rectangles obtained by cutting will be equal.□

75. example. *The length of the side of a square is 1. There are 19 points marked inside the square. These points and the vertices of the square are*

coloured red. No 3 of red points are on the same line.

Prove that there is a triangle with area not exceeding $\frac{1}{40}$ *all of whose vertices are red.*

Solution. Let us show that it is possible to divide the square into 40 triangles with red apexes without common inner points. As the sum of their areas is 1, then at least 1 area does not exceed $\frac{1}{40}$.

We shall begin to draw the segments connecting 2 red points. We draw each following segment so that it does not intersect any of the previously drawn. We are continuing this process so long that it is no more possible to draw any segment. At this moment the whole square is divided into triangles (if there was some part with a bigger number of sides - a quadrangle, pentagon etc. - then, by drawing one of its diagonals, it would be possible to continue to draw the segments). Besides, the full angle around every inner red point has divided into the angles of triangles (if it were not so, then it would be still possible to draw segments from this point). The sum of all angles of all the triangles consists of 19 full angles around the inner points and 4 right angles of the square. Therefore, their sum is $19 \cdot 360^{o} + 360^{o} = 20 \cdot 360^{o}$. As the sum of the angles of 1 triangle is 180^{o}; then there are $\frac{20 \cdot 360^{o}}{180^{o}} = 40$ triangles -this is what we had to prove.□

Now we shall discuss the problems connected with the conception of the magnitude of the angle. Here it is important to know 3 facts:

a) the sum of internal angles of the n-gon is $180^o \cdot (n-2)$,

b) the sum of external angles of a convex n-gon (taking one external angle at each apex) is 360^o,

c) if we divide the angle into several angles with rays leading out of the apex, then the sum of magnitudes of the separate parts is identical with the magnitude of the whole initial angle.

76. example. *What biggest number of acute angles can be in a 12-gon?*

Solution. As you see in Fig. 48, 9 acute angles are possible. We shall motivate why more such angles are not possible. Let us presume that 10 angles of the 12-gon are acute. Then the sum of magnitudes of these 10 angles is smaller than $10 \cdot 90^o = 900^o$. Each of the other 2 angles is smaller than 360^o; therefore the sum of magnitudes of all the angles is smaller than $900^o + 2 \cdot 360^o = 1620^o$. But the sum of the angles of a 12-gon is $180^o \cdot (12-2) = 1800^o$. We get a contradiction, so our presumption is wrong, and there cannot be 10 acute angles.□

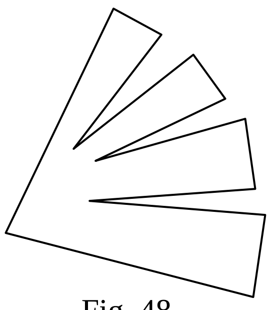


Fig. 48

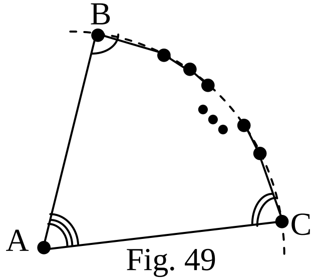


Fig. 49

77. example. *What biggest number of acute angles can be in a **convex** 12-gon?*

Solution. As can be seen in Fig. 49, it can be 3 (the acute angles are marked by small bows, but the nodes of the obtuse angles are situated on the arc of the circle between B and C). If a convex 12-gon had 4 acute angles then all their external angles would be obtuse (i.e. they would exceed 90^o), and the sum of magnitudes of obtuse angles would exceed 360^o - a contradiction. So, the presumption is wrong and there cannot be 4 acute angles.□

78. example. *In an 11-gon no two diagonals are parallel. Prove: among the straight lines holding the diagonals of this 11-gon it is possible to find such 2 which form an angle smaller than* 5^o.

Solution. You can draw 8 diagonals from each vertex of the 11-gon. So, in total there are $11 \cdot 8 = 88$ ends of diagonals; there are $88/2 = 44$ diagonals. We draw 44 straight lines through some point O, and each of them is parallel to another diagonal. The full angle round the point O divides into 88 angles, which are alternate angles in pairs. If none of them were smaller than 5^o, then their sum would not be smaller than $88 \cdot 5^o = 440^o$. However, this sum is 360^o. So, one of the angles formed in point O is smaller than 5^o; the same kind of angle is between the straight lines holding the respective diagonals.□

79. example. *On the straight line t there are 6 segments without common points. Each of them serves as the base where an equilateral triangle is*

constructed /designed/ (all on one side of the straight line t). We construct 6 circles whose centres are those apexes of these triangles which do not belong to the straight line t, but the radiuses are identical with the lengths of the sides of the respective triangles. Prove that there is not a point that belongs to all these circles simultaneously.

Solution. Let us presume that P may be such a point and ABC - one of the 6 triangles discussed (Fig. 50).

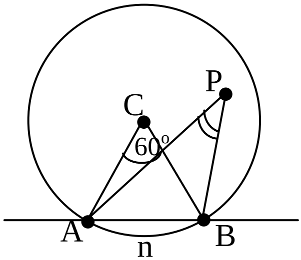


Fig. 50

As arc AnB corresponds to the angle ACB of the centre whose magnitude is 60^o, then $\smile AnB = 60^o$. As $\angle APB$ is inscribed (if P belongs to the circumference) or internal (if P is inside the circumference) angle that rests on arc AnB, then $\angle APB = 0.5 \smile AnB$, resp. $\angle APB > 0.5 \smile AnB$. In any case $\angle APB \geq 30^o$.

If point B belonged to all the 6 circles, then there would be $\alpha_1 \geq 30^o, \alpha_2 \geq 30^o, \cdots, \alpha_6 \geq 30^o$ (see Fig. 51). But it cannot be like that because apparently $\alpha_1 + \alpha_2 + \ldots + \alpha_6 < \angle MPN = 180^o$. We get a contradiction, so P cannot belong to all 6 circles to be discussed.□

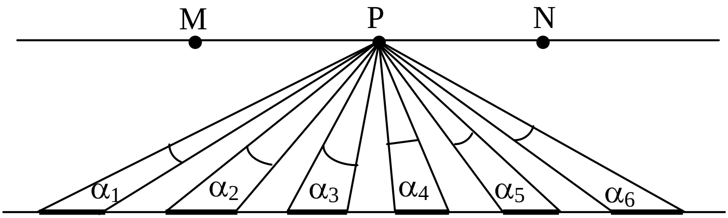


Fig. 51

Commentary. In order you did not imagine that the circles constructed in the way discussed, can generally split only by twos, or, at most, by threes, we advise you to make the drawing independently, where analogically constructed **5** circles **have** a common point (it can be done).

The object of summarizing may be the angular magnitudes of the arcs, too.

80. example. *A square and a regular triangle are inscribed in the same circumference. Their vertexes divide the circumference into 7 arcs. Prove that at least the size of one arc does not exceed* 15^{o}.

The first observation. If 7 arcs arise, then no square vertex coincides with any apex of the triangle.

Commentary. If we just observed that the sum of angular magnitudes of 7 arcs is 360^{o} and if at once we tried to estimate the smaller item only on the basis of this information, we would draw a conclusion that it is possible to find an arc that does not exceed $\frac{360^{o}}{7} = 51.42^{o}\ldots$ But we need a more powerful result.

Solution. First we shall discuss 3 arcs into which the triangle apexes divide the circumference; 4 vertexes of the square are placed along them. So, 2 of them belong to the same 120^o arc (Dirichlet principle **D_1**!). Let us presume that they are M and N (see Fig. 52) which are on the arc marked off by triangle apexes A and B. We observe that $\smile AB = 120^o$ and $\smile MN = 90^o$, therefore $\smile AM + \smile NB = 30^o$. From the equality $\smile AM + \smile NB = 30^o$ it follows that either $\smile AM \le 15^o$ or $\smile NB \le 15^o$ which we had to prove.

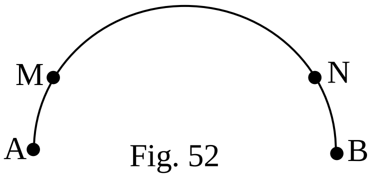


Fig. 52

VIII.4.2. SUM ESTIMATIONS AND OVERLAPPING

It is easy to understand that the following statements are correct.

Theorem about overlapping on the straight line.

Let us presume that given is a segment with length t. If in it there are located several segments whose sum of lengths exceeds t, then some segments overlap one another somewhere. Let us presume that given is a segment with length t. If we locate in it several segments whose sum of lengths exceeds $k \cdot t$ (k is a natural number), then there exists a point

which is covered by at least k + 1 of the located segments.

Theorem about overlapping in the plane.

Let us presume that given is a room with area S. If we put several not-folded up carpets into it, whose sum of areas exceeds S, then some carpets overlap somewhere. Let us presume that given is a room with the area S. If we put into it several not-folded up carpets, whose sum of areas exceeds $k \cdot S$ (k is a natural number), then there exists a point which is covered by at least k + 1 carpet.

The above mentioned theorems are geometric versions of $\mathbf{D_4}$ and $\mathbf{D_5}$ theorems mentioned at the beginning. We advise you to prove them independently, as well as formulate and prove similar statements about what happens if the sum of longitudes of the located segments is smaller then t, i.e., smaller then $k \cdot t$, or if the sum of areas of located carpets is smaller than S, i.e., smaller than $k \cdot S$.

81. example. *In the square ABCD where the length of the side is 1, several circumferences are situated whose sum of lengths is 20. Prove that it is possible to draw a straight line that is perpendicular to AB and intersects at least 7 of these circumferences.*

Solution. We shall project all the circumferences on the side AB (see Fig. 53). Each circumference projects in a segment whose length is equal to its diameter. If the diameters of the circumferences are $d_1, d_2, \ldots, d_n$, then the sum of their lengths is $\pi \cdot d_1 + \pi \cdot d_2 + \ldots + \pi \cdot d_n = 20$.

Consequently, $d_1 + d_2 + \ldots + d_n = 20/\pi > 6$. Therefore in the segment AB, whose length is 1, n segments are located whose sum of longitudes exceeds 6. Therefore at least some point of AB is covered by no less than 7 segments (if every point of AB were covered by no more than 6 segments, then their common length could not exceed the six fold length of AB - a contradiction). Drawing through this point Q a straight line perpendicularly to AB it intersects all those (at least seven!) circumferences, whose projections cover the point Q. The problem is solved.□

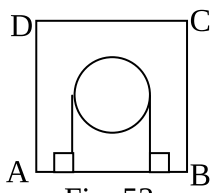


Fig. 53

82* . example. *In the circle with the radius 10 there are 122 points (inside or on the margin). Prove that among them it is possible to find 2, which are on the distance smaller than 2 from each other.*

Solution. Let us draw a small circle with radius 1 around each point. All such small circles are located inside the broken circumference (with radius 11) or they contact it inwardly (see Fig. 54).

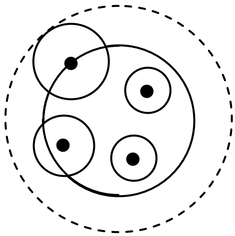


Fig. 54

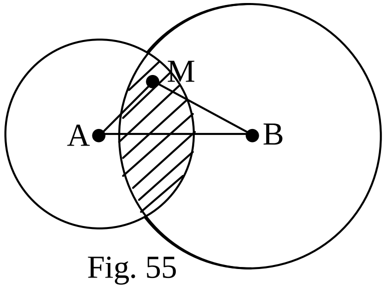


Fig. 55

The broken circumference bounds the circle with the area $\pi \cdot 11^2 = 121\pi$; the area of each small circle is $\pi \cdot 1^2 = \pi$. As the sum of small circles' areas exceeds the area of the "broken" circle, then some 2 small circles overlap each other. But then the distance between their centres is smaller than 2 (see Fig. 55): $AB < AM + MB \le 1 + 1 = 2$, consequently $AB < 2$ what we had to prove. The problem is solved.□

We can use similar arguments in all problems where the demand is to locate the figures so that no 2 figures are nearer each other than at a distance d. Then we extend each figure by a $\frac{d}{2}$ wide "edging"; the extended figures must not overlap one another.

VIII.4.3. SUM ESTIMATIONS AND THE TRIANGLE INEQUALITY

Alongside with the triangle inequality (in each triangle ABC the inequalities AB + BC > AC, AC + CB > AB, BA + AC > BC hold) it is suitable to use the following facts (they both follow from the triangle inequality) in solution of the problems of this group.

Quadrangle inequality.

In the convex quadrangle ABCD the inequalities AC + BD > AB + CD and AC + BD > AD + BC hold (the sum of diagonals' lengths exceeds the sum of the lengths of 2 opposite sides).

▼ Proof. If the diagonals intersect in point O, then (see Fig. 56) AO + OB > AB and OC + DO > CD.
Adding up these inequalities we get $(AO + OC) + (DO + OB) > AB + CD$ or $AC + BD > AB + CD$.
The second inequality is proved in the same way. ▲

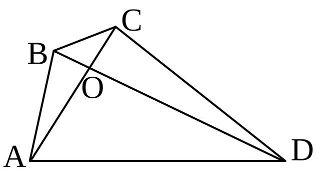


Fig. 56

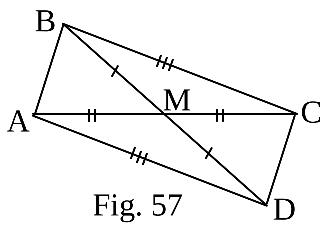


Fig. 57

Theorem about the length of the median.

The median of triangle is shorter than half the sum of those sides between which it is situated.

▼ Proof. We extend ABC up to the parallelogram ABCD. The point of intersection of its diagonals is marked M (Fig. 57). Then M is the midpoint of AC and BD. So BM is the median of ABC. From the triangle inequality, applying it to ABD, we get AB + AD > BD. But $AD = BC$ and $BD = 2 \cdot BM$, therefore from this inequality follows $AB + BC > 2 \cdot BM$ or $BM < \frac{1}{2}(AB + BC)$ what we had to prove. ▲

83. example. *Prove that in a convex quadrangle at least one diagonal is longer than one-fourth of the perimeter.*

Solution. In accordance with the quadrangle inequality, in a convex quadrangle ABCD there we have $AC + BD > AB + CD$, $AC + BD > AD + BC$ (see Fig. 56).

Adding up these inequalities, we get

$2(AC + BD) > Per(ABCD)$ or $AC + BD > \frac{1}{2}Per(ABCD)$. (*)

From here it follows that either AC or BD exceeds $\frac{1}{4}Per(ABCD)$; if it were both $AC \le \frac{1}{4}Per(ABCD)$ and $BD \le \frac{1}{4}Per(ABCD)$ then, by adding up both the last inequalities, we would get $AC + BD \le \frac{1}{2}Per(ABCD)$ that contradicts (*).□

84. example. *The length of the diagonal of the square ABCD is d. Inside it a point M is taken. Prove that the distance of M to at least 2 vertexes is not smaller than* $\frac{d}{2}$.

Solution. In accordance with the triangle inequality, for each point M we have $AM + CM \ge d$ (equality is possible only if M is on AC). If the sum of 2 items is not smaller than d, then at least one of them is not smaller than $\frac{d}{2}$. The same way it is proved, that the distance of M either to B or D is not smaller than $\frac{d}{2}$.□

85*. example. *On the table there are 100 exactly functioning clocks. Prove: no matter what the point P of the table may be, a moment will turn up, when the sum of distances from P to the centres of faces of the clocks will be smaller than the sum of distances from P to the ends of the minute hands.*

Solution. We will choose such a moment when none of the straight lines, where the minute hands are situated, leads through P. We shall mark the positions of the ends of minute hands at this moment by $A_1, A_2, \ldots, A_{100}$, but their positions half-an-hour later by $B_1, B_2, \ldots, B_{100}$. The choice of the moment guarantees that point P forms **a triangle** with each pairs of points (A_1, B_1), (A_2, B_2) etc. (see Fig. 58). We shall mark the corresponding centres of faces by $O_1, O_2, \ldots, O_{100}$.

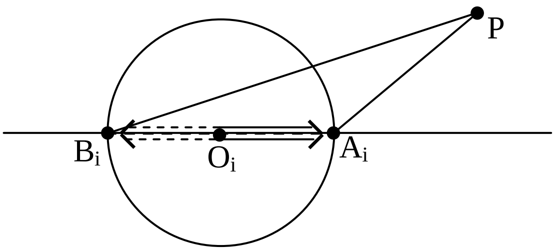


Fig. 58

Then PO_1 is a median in the triangle PA_1B_1, PO_2 is a median in the triangle PA_2B_2 etc. In accordance with the theorem about the length of median we get

$$PA_1 + PB_1 > 2 \cdot PO_1$$
$$PA_2 + PB_2 > 2 \cdot PO_2$$
$$\ldots$$
$$PA_{100} + PB_{100} > 2 \cdot PO_{100}.$$

By adding up these inequalities and grouping the members we get

$$(PA_1 + PA_2 + \ldots + PA_{100}) + (PB_1 + PB_2 + \ldots + PB_{100}) > 2 \cdot (PO_1 + PO_2 + \ldots + PO_{100}).$$

From here it follows that either

$$(PA_1 + PA_2 + \ldots + PA_{100}) > (PO_1 + PO_2 + \ldots + PO_{100}),$$

or

$$(PB_1 + PB_2 + \ldots + PB_{100}) > (PO_1 + PO_2 + \ldots + PO_{100}).$$

Consequently, either the moment chosen at the beginning or the moment following after half-an-hour is useful.□

VIII.5. SOME PECULIAR METHODS OF "HUTCH" CONSTRUCTION

Certainly, here it is not possible to comprise all the methods formed till now; besides, we must realize that there are many more peculiar "inventions" to be discovered. We shall show some striking examples.

86*. example. *A regular 12-gon has 6 vertexes painted white, but the other 6 - black. Prove that it is possible to find 2 identical quadrangles: one of them has only white vertexes, but the other - only black vertexes.*

Solution. We shall discuss two copies of this 12-gon which are placed exactly one above the other. The lower 12-gon is not altered. We shall discuss all the 12 possible positions the upper 12-gon may have if it is turned by 30^o 12 times in succession (the vertexes of both 12-gons coincide in pairs in each of these 12 positions). In all the 12 positions together, at times every vertex of the upper 12-gon coincides with every vertex of the lower 12-gon.

As there are 6 white vertexes in the upper 12-gon and 6 black in the lower, then the coincidence of the white upper vertex with the black lower vertex happens $6 \times 6 = 36$ times. In one position - after the 12^{th} turn - there is no such coincidence, because the upper 12-gon has returned to the initial position and the white vertexes coincide with the white, but the black - with the black ones. So, the above mentioned coincidences are distributed among 11 positions of the upper polygon. As $36 > 11 \cdot 3$, then in one of the positions at least 4 upper white vertexes coincide with the lower black ones. These coincidences determine both the necessary quadrangles.□

87^k. example. *Given are 13 different rectangles. The lengths of their sides are expressed by whole numbers. None of the sides is shorter than 1 and none is longer than 12. Prove that among them it is possible to choose such 3 rectangles, that the first one can be completely overlapped by the second, and the second one - by the third rectangle. (By overlapping, the rectangles must be placed so that their sides are parallel).*

Solution. Let us look at the table of Fig. 59.

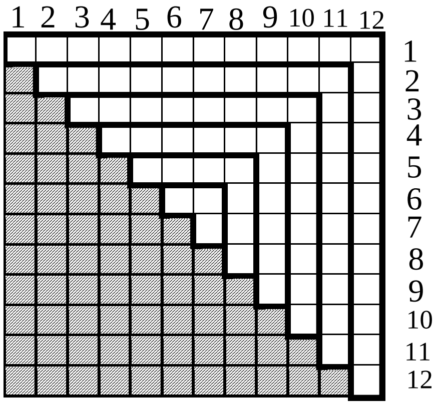


Fig. 59

Depending on its parameters, it is possible to put each rectangle exactly into one of the white squares. We can see the distribution of these squares into 6 “hutches”.

As $13 > 6 \cdot 2$, a “hutch” may be found where at least 3 rectangles are placed.

It is not difficult to understand: if in 1 “hutch” 3 rectangles follow one after the other (first comes A, then B and then C), then it is possible to overlap rectangle A by rectangle B, but B - by C. The problem is solved. □

88. example. *A square consisting of 6 x 6 cells is made up of 18 rectangles 1 x 2 cells each (we shall call these 1 x 2 cell rectangles dominos). Prove that it is possible to divide this square into 2 rectangles without cutting any domino.*

Solution. Let us presume that it is not possible to do it. Then each of the 10 straight lines shown by Fig. 60 intersects **at** least 1 domino (otherwise we could divide the square along this straight line).

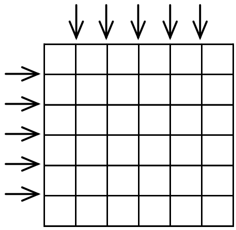


Fig. 60

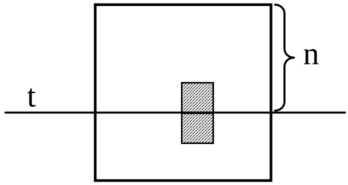


Fig. 61

We shall apply Dirichlet principle in quite unusual way. Let us prove: it is impossible that **all straight lines intersect more than 1 domino simultaneously**. Really, if each of 10 straight lines intersected at least 2 dominos, then there would be at least $10 \cdot 2 = 20$ dominos in total (it is clear that only 1 straight line can intersect each domino); but there are only $\frac{36}{2} = 18$ dominos. We conclude that among the described straight lines there is one that intersects **exactly 1 domino**. We shall presume that it is the straight line t (Fig. 61). It is not possible. Really, in the part of the square above the straight line t there are $6 \cdot n$, i.e., an even number of squares. These 6n squares consist of a whole number of uncut dominos and 1 black square in the upper part that comes from **the only** cut domino. Consequently, the number of squares in the upper part is "the even quantity + 1", i.e., an odd number. We get a contradiction. The problem is solved.□

89. example. *36 points are located like a quadratic grid (see Fig. 62). What smallest number of straight lines must be drawn, so that they divided the plane into such regions, that each of them contained no more than one point (or, what smallest number of straight lines is enough to separate all the points from each other)?*

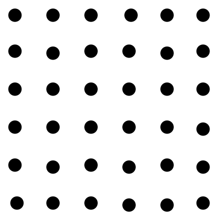


Fig. 62

<u>Apparent observation</u>: it is enough to draw 5 vertical straight lines (one between every two columns of points drawn next to each other) and 5 horizontal straight lines; 10 altogether. Lasting attempts do not bring a better result (a smaller number of straight lines).

<u>The attempt of the optimality proof</u>. A natural idea is to try to prove, that, by drawing 9 straight lines, definitely there appear less than 36 regions. If it were the case, then, according to Dirichlet principle, in the situation of 9 straight lines there would be more than 1 point in some region. Unfortunately, it turns out that the number of regions can reach and even exceed 36.

As seen in Fig. 63, 3 straight lines can divide the plane into 7 regions. If we draw the fourth straight line so that it intersects all 3 preceding ones in different points, on the fourth straight line there appear 3 points of intersection. Therefore it splits into 4 parts (2 rays and 2 segments). Each of these 4 parts divides one of the old regions into 2 parts, so the number of regions increases by 4. We can show similarly that drawing the fifth line can increase the number of regions by 5, drawing the sixth line - by 6 etc. So, 9 straight lines **can** divide the plane into $7+4+5+6+7+8+9=46$ regions, which would be as if more than enough for locating 36 points one by one.

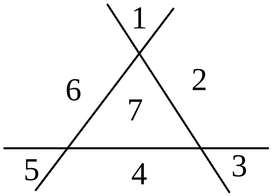


Fig. 63

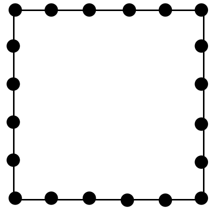


Fig. 64

So we must look for another way of solution.

Proof of the optimality. We shall discuss only the **external** of the given 36 points (see Fig. 64); they all, too, must be separated one from the other. It is possible to separate 2 alongside points only by intersecting the segment that connects them.

There are 20 segments altogether and each straight line can intersect no more than 2 of them. Consequently, the number of the straight lines must be at least $\frac{20}{2}=10$. The problem is solved.□

Commentaries.

1. We have the same situation as in the solution of the example 13 - it seems that it is possible to separate easier only the external points than all of them. However, exactly the analysis of this “easier” case allows us to solve simultaneously the general problem, which did not come easy before.

2. Like in the solution of example 66, we see: if the proof does come not easy when we use “hutches” and “rabbits” of one kind (parts of the plane and points), it may come easy if we choose other "hutches" and "rabbits" (in this case "hutches" are the drawn straight lines, but "rabbits" - segments to be intersected; if we draw only 9 straight lines, then one of them should intersect 3 segments, which is not possible).

3. The idea to discuss the "external" points is not as accidental as it may seem. Many people admit that, for example, the nature of a man shows itself in extreme situations. Like in mathematics, it is quite often that you succeed if you discuss the "extreme", "the most outstanding" elements from some point of view - the biggest among the numbers discussed, the most distant of the points discussed, the triangle with the smallest area among the discussed ones etc. This kind of reasoning has its own title - "the method of extremal element". We may say that we already applied the method of extremal element in the solution of example 27: **the smallest** prime multiplier of each number was discussed.

We could say that, in a hidden way, the application of Dirichlet principle is almost always connected with seeking of the extremal element: really, usually it is possible to choose as wanted "hutch" the one, which contains the most of "rabbits" (or the least - depending on the kind of Dirichlet principle applied).

Problems for Independent Solution.

113. The length of the side of an equilateral triangle is 2. Five points are marked in the triangle. Prove that some 2 of them are situated at the distance not exceeding 1 from each other.

114. The length of the side of an equilateral triangle is 10. There are marked 201 points in it. Prove that it is possible to find such 3 points among them, all of which are at the distance not exceeding 1 from one another.

115. The dimensions of a square are 8 x 8; four points are marked in it. Prove that it is possible to find such 2 marked points which are at the distance not exceeding $\sqrt{65}$ from each other.

116. Each contour point of a regular triangle is coloured white or black. Prove that it is possible to find a right-angled triangle that has all the apexes painted in the same colour.

117*. Each point of a plane is painted black or white. Can you definitely find an equilateral triangle that has all its apexes painted in the same colour and the length of its side 1m?

118. A regular triangle is divided into 16 smaller equal triangles (Fig. 65). Every apex of the triangles of the grid is painted white or black. Prove that it is possible to find an equilateral triangle whose all apexes are painted in the same colour.

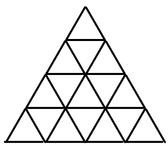


Fig. 65

119°. Out of 40 vertexes of a 40-gon 17 are painted red. Prove: it is possible to find an isosceles triangle with all apexes painted red.

120°. 10 x 10 points are placed in the shape of quadratic grid. What is the smallest number of segments for a broken line (not obligatory closed) that goes through all the points?

All segments are either horizontal or vertical.

121^k. Solve example 71 if you withdraw the demand, that all the segments be horizontal or vertical.

122^o. 5 straight lines intersect mutually. What is the biggest possible number of sides of a polygon, whose all sides are situated on these straight lines?

123. Can a 1000-gon have 501 parallel sides?

124^o. A square consists of 8 x 8 cells. It is cut into 13 rectangles. The cuts run only along the border lines of the cells.

Prove that there are 2 identical rectangles among those formed by cutting.

125^o. Inside the triangle there are 10 red and 20 blue points; also the apexes are red. No 3 coloured points are situated on one line. Prove: it is possible to find a triangle, whose all apexes are red and which does not contain any blue point.

126^o. What biggest number of acute angles can be

a) in an octagon,

b) in an eleven-gon?

127^o. What biggest number of acute angles can be in a convex 10-angle?

128. The circles are constructed on the sides of a convex quadrangle as diameters. Prove that the circles cover all the quadrangle.

129*. Prove that in every polygon, having at least 4 vertexes, it is possible to draw a diagonal that does not lead out of this polygon (we presumed it in the solution of example 75).

130°. Prove: among the diagonals of each 10-gon it is possible to find either 2 parallel diagonals, or such 2 diagonals, where the angle, formed by their holding straight lines, is smaller than 6^{o}.

131°. A regular 9-gon and a regular 10-gon are inscribed in the same circumference; their vertexes divide the circumference into 19 arcs. Prove that the size of at least one arc does not exceed 2^{o}.

132. What smallest number of triangles can cover a convex 12-gon? The triangles may overlap mutually, but they must not lead outside the 12-gon.

133*. The length of both diagonals of a convex quadrangle is 20. Prove that at least one side is longer than 14.

134°. The length of the side of a square is 1. Inside the square there are several circumferences, and the sum of their lengths is 30. Prove that it is possible to draw a straight line that intersects at least 10 of them.

135*. In the circle with the radius 1 there are several segments, whose sum of lengths is 32. Prove that it is possible to draw a straight line that intersects

a) at least 8,

b) at least 9 segments.

136*. Given is a circle with radius 2. What smallest number of circles with radius 1 must be drawn, so that the initial circle is completely covered?

137°. Prove: in the circle with radius 14 it is not possible to mark 226 points so, that the distance between every two of them exceeds 2.

138*. In a square, whose length of side is 1, 170 points are chosen.

Prove: it is possible to find two of them within the distance not exceeding 0.009.

139. Inside the square with side length 15 there are 20 small squares with side length 1 each.

Prove that inside the "big square" it is possible to place a circle with radius 1 having no common point with any of the small squares.

140^{o}. On the plane there are given 10 points and a circumference with radius 1. Prove that it is possible to find such a point on the circumference, whose sum of distances to the 10 points mentioned before exceeds 10.

141. Five points with integer coordinates are given in the plane.

Prove: at least on one of the segments, connecting these points in pairs, we can find some other point with integer coordinates.

142*. Inside a convex 10-gon a point is marked, which is not situated on any diagonal. A hare is sitting there, but a hunter is standing on each vertex. All the hunters simultaneously fire at the hare, but at that moment the hare crouches down and the bullets miss him (we presume that the bullets do not hit one another). Prove that at least one side of the 10-angle will not be hit.

143. 13 straight lines are drawn; each of them divides the square ABCD into 2 trapezia, but the ratio of their areas is 1 : 2. Prove that at least 4 of these straight lines lead through the same point.

144^{k}. A closed broken line connects all the vertexes of a regular 20-angle. Prove that the line has at least 2 parallel segments.

145*. A piece of paper consists of 10 x 10 identical quadratic cells. Several squares are marked so, that simultaneously

> **a)** the sides of each marked square run along the borders of the cells,
>
> **b)** no marked square covers any other one completely.

What biggest number of the squares can be marked?

146. The sides and diagonals of a convex n-gon are passages of a labyrinth. What smallest number of electric bulbs must be put in the labyrinth to make it fully lighted? No 3 diagonals intersect in the same point.

147*. The shape of the room is a 30-gon. Prove that it is definitely possible to illuminate it with 10 lamps.

148. Devise a room in a shape of 30-gon, for whose lighting at least 10 lamps are necessary.

149. A square consists of 10 x 10 cells. What biggest quantity of the cells' centres is it possible to paint so, that no 3 painted points are on the same straight line?

150[k]. The inner parts of the sides of a regular triangle are mirrors. A beam of light is let inside it. The beam is moving in accordance with the rule "the angle of incidence is equal to the angle of reflection" (see Fig. 66). It is known that the ray has passed 7 times through some point. Prove that some time this ray will pass through this point again.

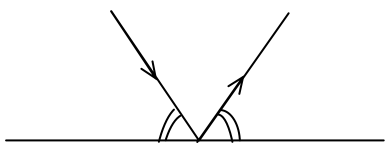


Fig. 66