The LAIMA series

Romualdas KAŠUBA

WHAT TO DO WHEN YOU DON'T KNOW WHAT TO DO?

R. Kašuba. What to do when you don’t know what to do?

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The book analyses psychological aspects of problem solving on the basis of contest problems for junior (4th – 9th Grades) students. Nevertheless, the approaches discussed are of value also for highest grades, for teachers, problem composers etc. The text can be used by all those who are preparing to research in mathematics and/ or to math contests.

The final version was prepared by Ms. Dace Bonka.

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To Ieva - a daughter and a friend.

The author.

FOR THE ENGLISH EDITION

The author would like to express his gratitude to the organizers and publishers of LAIMA series for publishing Part I of Lithuanian manuscript in English. It is a great joy and pleasure for me.

I am especially thankful to Professor Agnis ANDŽĀNS for his constant help and delicate support. Professor Andžāns was also the first official reviewer of the Lithuanian edition. Without his inspiration and practical care my translation wouldn't be made and the volume wouldn't be published.

Romualdas Kašuba

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ABOUT LAIMA SERIES

In 1990 international team competition "Baltic Way" was organized for the first time. The competition gained its name from the mass action in August, 1989, when over a million of people stood hand by hand along the road Tallin - Riga - Vilnius, demonstrating their will for freedom.

Today "Baltic Way" has all the countries around the Baltic Sea (and also Iceland) as its participants. Inviting Iceland is a special case remembering that it was the first country all over the world, which officially recognized the independence of Lithuania, Latvia and Estonia in 1991.

The "Baltic Way" competition has given rise also to other mathematical activities. One of them is project LAIMA (Latvian - Icelandic Mathematics project). Its aim is to publish a series of books covering all essential topics in the area of mathematical competitions.

Mathematical olympiads today have become an important and essential part of education system. In some sense they provide high standards for teaching mathematics on advanced level. Many outstanding scientists are involved in problem composing for competitions. Therefore "olympiad curricula", considered all over the world, is a good reflection of important mathematical ideas at elementary level.

At our opinion there are relatively few basic ideas and relatively few important topics which cover almost all what international mathematical community has recognized as worth to be included regularly in the search and promoting of young talents. This (clearly subjective) opinion is reflected in the list of teaching aids which are to be prepared within LAIMA project.

Eighteen books have been published so far in Latvian. They are also available electronically at the web - page of Latvian Education Informatization System (LIIS) http://www.liis.lv. As LAIMA is rather a process than a project there is no idea of final date; many of already published teaching aids are second and third versions and will be extended regularly.

Benedikt Johannesson, the President of Icelandic Society of mathematics, inspired LAIMA project in 1996. Being the co-author of many LAIMA publications, he was also the main sponsor of the project for many years.

This book is the fourth LAIMA publication in English. It was sponsored by the Scandinavian foundation “Nord Plus Neighbours”.

SOME INTRODUCTORY WORDS CONCERNING ELEMENTARY PSYCHOLOGICAL ASPECTS IN PROBLEM SOLVING

"The thing can be done," said the Butcher, "I think.
The thing must be done, I am sure.
The thing shall be done! Bring me paper and ink,
The best there is time to procure."

(Lewis Carroll, The Hunting of the Snark)

Why do the authors prepare and write texts? The public opinion would answer that question most probably with the words that the author is writing because he wants to tell and explain to us some things – this is the common truth.

What do we want to say and express? The answer is even simpler than that what was mentioned above.

The author would like using normal and for every person understandable words to discuss some simple problems and express – so my hope - some clear and accessible (possibly always optimistic but in no way or seldom mystic - if we would allow us an attempt to express ourselves in somehow funny way) ideas.

It is no secret that the effective thinking - which is being since many centuries first of all associated with mathematics – is accessible for us human beings not just in the same degree and consequently is beloved by us also not in the very same degree. There are persons who are not at all fond of it. It is understandable that it is impossible to be fond of the field where according to your opinion you are not successful enough or of the field where your neighbour is clearly better (you think so) than you are.

Is it possible to like a least a bit the matters which you can't master from the first attempt, something, that is not going well enough or even looks a bit frightening? In the same time from all sides we hear streams of words that the math is so valuable, precious and almost miraculous so to say full of wonders and gift of heaven. What can be so especially precious in all the formulas filled with lots of numbers and letters, what so extraordinary may be hidden in all these complicated drawings and long lined proofs? What can be so attractive in these matters which (even you and me) are not able to catch from the first thought and sight, view and glimpse?

At that place we can fell in some psychological trap: we can forget that if any thing is difficult for me then it is highly probably that would be difficult not only for me. If it is easy and that's way promising for me then it is very probably it would appear promising interesting for you as well.

The great British mathematician which name is Littlewood (rather remarkable name, isn't it?) once told about the test which he had written soon after entering the University remembering that he wasn't able to solve some 2 of proposed problem. Somehow occasionally he saw that his neighbour marked one of these 2 problems as a solved problem. Some minutes later Littlewood was able to mark that problem as a solved one too. We hope that the reader is not thinking that Littlewood have seen also his neighbour's solution.

We would like to start from simplest thoughts and everyday’s analogies. There is also famous Chinese aphorist saying that the traveller overcomes the road starting with the first step.

Now to the following problem which is clearly not only of gastronomical importance. If anybody would go on explaining us that the preparing of tasty dishes is a thing that could be highly recommended who could have anything against it? We could add that this is also important, of great value and importance for the whole gastronomy industry as well, so influencing our moods, way of life and general progress of the mankind.

It can be no doubt about it.

We would listen patiently, with remarkable pleasure and attention that to do something well enough is so important; we would listen for hours especially if we’ve right after our meals. But after that careful listening we will very probably start to feel that we are missing something. It is rather strange feeling.

We would feel that if anybody is so perfectly explaining to us – we are not joking now - how nice it is to prepare tasty dishes so he could also show us at least for the sake of completeness how practically to do and realize that ideas. Then we could also state that these dishes are really more tasteful as what we are suggesting and preparing.

Well, you would say that for all this we would need a kitchen, pots, food supplies, time and patience and even exotic species.

Exactly the same thing is with these suggestions and remarks concerning how useful and important it is to solve mathematical problems and what a great value in life have the abilities of adequate reasoning and exact thinking.

We agree that with these words about tasty meals and valuable dishes it is also possible to achieve a lot e.g. to wake an appetite in you and me or the desire to start preparing it.

Still we resolutely remain by the fundamental thought that the best way make us to believe that tasty soap is the force is to prepare that soup in our presence right now with an immediate proposal to taste it and frankly report whether you liked it.

It is similarly with the problem solving – independently of what kind – let them be the every day's problems, let it be mathematical tasks or even our future perspectives, sights and ideas.

Common sense and life experience convinces us that

The best way to make me believe that to produce nice subjects is beautiful is either to produce exactly the same or similar subject in my presence or to propose me to produce something similar or even more valuable and better. That would mean also that you have a confidence in me. If I feel that you indeed have a confidence in me then my wish to do what you would suggest me to do is remarkably higher.

At this place discussing about what can be "higher" we are citing for a fun a limerick:

There was a new servant maid, named Maria,
Who had some troubles with lighting the fire.
The woods being green
She used gasoline
Her position by now is much higher.

We would like to ensure the reader that the problem solving will bring in “higher position” in the sense that we will always accept and enjoy.

Problem solving isn’t dangerous.

Let us also try to act in a similar way, proposing considering and discussing some simple(st) but in the same time useful and accessible problems with the attempts of showing and demonstrating how they could be mastered. Let’s act together applying mutual care, support and advices.

This is the first part of the author’s manuscript [9], which appeared in Lithuanian in 2005. The author would be extremely content of any remark, comment and of any kind of the reader opinions.

The first printed Lithuanian book is a catechism of Martynas Mažvydas (Martinus Mosvidius). The very first words of that catechism are: “Take(th) me in your hands and read(eth) me and doing that understand(eth) me”.

The author isn’t so sure whether he succeeded in translating in sufficient degree into English these old Lithuanian words as well as the whole text of itself, but he categorically believes **that the best way to learn how to solve problems is to regard them trying to understand what they are about and how the tasks raised by them could be achieved, developed and generalized.**

One could imagine that each at least a bit interesting problem is also an invitation and a probe of our mind’s power, character and possibilities as well.

To these experiences, thanks God, we do not need any complicated tools or special circumstances – only sheet of paper, pen and bright head with a bit persistence and (sometimes lot of) patience.

Not in vain the people believe that there are no means, which could stop a persistent person from realizing its intentions.

We will start with a problem, which was proposed in an open Latvian contest for grade 5 or 6.

At that place we would like to express our opinion about the possible attitude towards the so-called problems of grade 5. Please do not believe that you can always solve them in a minute even if you are after the University education. They cannot be so simple; they are only accessible for all who have bright head. They usually demand practically no instrumental knowledge or long formulas. Otherwise they could be not so simple or even a bit complicated similarly as pupils of grade 5 themselves.

We even dare to remind you perhaps the most important psychological law of effective and successful communication could be formulated as follows:

DON'T THINK ANOTHER PERSON BEING LESS CLEVER THAN YOU ARE.

CHAPTER I.
MULTIPLES OF 7 WITH SOME LATVIAN FLAVOR

All the problems that we intend to solve and discuss will be of such a kind that they are accessible to everyone who is eager to achieve some progress or understand more than one was able to understand before and who has a least a bit time and patience for that.

Some of these problems will be quite easy others perhaps would appear more difficult or even mysterious especially from the first point of view but all of them will be such that for the solving of them no special knowledge, especially such exceeding the usual average school material, is needed.

In other words we propose simply looking but, we hope, essentially interesting problems and tasks and we will try to teach a reader how to deal with them.

The main psychological problem is to convince the reader that it is always possible to achieve something – alone the understanding what the problem is dealing with is very important, valuable and precious.

The author is going to repeat some well-known matters - for the sake of the reader he is ready to do it 10 times if necessary.

Firstly you are advised to read the text of the problem with some concentration. If after that you still do not know what to do, read the text of the problem once again. If again you are not at all sure what to do, read the text of the problem for the third time. If even now you do not know what to undertake, don’t read the whole text any more but only what the problem asks you to do, if necessary again three times.

After that if the problem remains for you not understandable enough we advise to do anything that is least slightly connected with the given problem.

In no way be ashamed to do smallest things if you feel that they are connected with problem you are dealing with.

Don’t be astonished to experience that some of these simply looking and really accessible problems were proposed in the mathematical contest of the highest range including even the International mathematical Olympiad that is, simply speaking, World cup of modern elementary – and sometimes not especially elementary – mathematics.

Sometimes there are really – who could only believe it - only few steps providing from the average school problem to the task representing highest Olympiad level – the reader will see it with his own eyes.

Don’t be astonished to lay the problem aside if you feel that it is necessary and don’t be in any case afraid to return to the given problem again (and again) – you are able to achieve much more than you dreamed or imagined.

You need only to take more time and even perhaps slightly more patience.

There are situations when the highest professional differs from the amateur only by the circumstance that the professional knows only one small fact more and otherwise they are absolutely equal.

But this small thing may be exactly the last stroke which breaks the camels – or in our case, problems – back.

But we are of course also able to notice and master the small thing that turned out to be so important.

There is an aphorism: **not (only) the Saints are these who form the pots**.

And in general no one doubts that to solve the problem or to seek the truth is essentially the same.

Now we are passing to the promissed Latvian problem. Before starting this we intend to grade or to structure it starting from simplest almost obvious remarks and moving to more complicated matters. The problem in question was discussed also in [1].

Let us regard any positive integer who is divisible by 7 and add all its digits. If such a divisible by 7 number is 147 (indeed, $147=7\cdot21$), then the sum of its digits will be

$$1+4+7=12.$$

We are dealing so often with the sum of digits of the positive integer n that we employ the special name for such sums, namely, we denote it by S(n).

These are the first simplest questions concerning the sums of digits of numbers that are multiplies of 7.

1. Can the positive integer who is divisible by 7 or, in other words, multiply of 7 have the sum of digits equal 10?

2. Can such a sum be equal to 100?

Other questions about such sums are a slightly more complicated or so to say no more as easy as possible:

3. Can the sum of digits of a multiply of 7 be 2007?

4. What is lowest possible sum of digits of the multiply of 7?

And finally the most complicated or almost philosophical question:

5. Which positive integers are the sums of digits of a multiplies of 7 and which are not?

In this place let us make some digression of philosophical nature – because of you can't find non-trivial or at least a bit interesting problem which wouldn't impulse or awake some psychological problems or difficulties as well: to deal with a problem is always very instructive or - using the nowadays terminology - challenging that is interesting, meaningful and useful.

So what are the psychological aspects that arise firstly right now?

We see at once that the problem is graded just as we intended.

At the first step we are asked something which is so simple almost obvious in such a degree that asking that in the same time we could beg your pardon for bothering you with such a simple things.

It slightly similar situation as if on the street a passer-by would ask you if your mother is your relative or not?

Why is it being done?

Perhaps the reason is that the passer-by wishes to involve us into an action or even awake our ambition, fantasy or, shortly speaking, to create the situation which involves our thinking capacities or the power of mind.

It's so well known and everyday confirmed that our thinking capacities, fantasy and ambition are so subtle instruments of our human nature. If you succeed in arising them the effects of this lasts so long – days and years and sometimes even our whole human life.

That is true under one condition: we ought to feel a clear desire make some progress be ready to do something and strongly believe that we are able to master the situation.

Perhaps not necessarily right now or in few minutes but surely after some time.

So let us start answering these simplest questions – let us get involved into promising action of solving or seeking the truth.

So once again: it possible or not that the sum of digits of a multiply of seven be equal 10?

Let us begin with experiment part or otherwise let us regard the very first multiplies of seven that is let us look to the numbers

7, 14, 21, 28, 35, …

We immediately state that with a number we were asked in the first or "involving" part of our problem we are already done: the number 28 is such a number we've asked for because the sum of digits of 28 is exactly 10.

In whole world you would hardly find anyone who would doubt this.

Now we could continue regarding these multiplies of 7 or regard the numbers

42, 49, 56, 63, 70, 77, 84, 91, 98, …

We see that the sums of digits of all these first multiplies are

7, 5, 3, 10, 8, 6, 13, 11, 9, 7, 14, 12, 10, 17, …

that is we could state that though these sums are not strictly increasing, but in general we are somehow clearly convinced that they are growing.

On the other hand it could be noticed that this growth follows rather slowly – in such a manner we'll achieve hundred as a sum of digits not sooner than after good half an hour of careful writing of these consecutive sums. In the same time we do not forget that the second number that is waiting for attention after clearing whether 100 can be the sum of digits of multiply of seven, a slightly bigger number 2007 which also called "a number of the year" is waiting for us with the same question.

The answer is a case of 100 is yes. Now it appears useful and possible to use the idea of putting some numbers with some properties together in order to form one bigger number with the same or similar properties.

From the technological point of view we simple write some numbers "together". It is also possible that we write down one and the same number several times and then we can "put all these copies of the same number together".

Putting together two times the number 28 we will of course get the number 2828. How do you think: after putting together two copies of 28 do we not lose the divisibility of the composed number by 7 or not?

Surely he don't because alone the slight remembering of the long division allows us immediately to state that similarly like $28=4\cdot 7$, so is $2828=7\cdot 404$ and $282828=7\cdot 40404$ etc. We clearly see that if any integer N is divisible by another integer M, then after putting any number of "copies" of N together we'll have that the composite number will be again divisible by M.

The modern reader thanks that calculator possess rather small experience with an addition or subtraction or multiplication of numbers by hand not to speak about long division. At least some experience with long division appears sometimes to be of remarkable value and importance. So it’s now in our case too when we try to explain that if we put together some copies of the integer M then any divisor of M remains the divisor of this composite number too.

Using that fact we state that all these numbers 2828, 282828, 28282828, … remain divisible by 7 alone from the fact that 28 is divisible by 7.

The sum of digits of 28 is $2+8=10$; the sum of digits of the number 2828 is already $2+8+2+8=20$. We need this sum be 100 that is we can put together 10 times the number 28 and so we can came across the number

$$28282828282828282828,$$

which we rather often write also as

$$28\,282\,828\,282\,828\,282\,828$$

grouping the digits into the blocks of three, counting from the right to the left..

Now of course it’s obvious that

$$28\,282\,828\,282\,828\,282\,828=7\cdot4\,040\,404\,040\,404\,040\,404.$$

Now we are finished with second part of our problem and we have no doubts concerning the fact that every integer which last digit is 0 is a sum of digits of some multiply of 7.

But now what about the number of the year 2007? His last digit is not 0. What to do? The idea is again the same that we can put together different numbers who are divisible by 7 without loosing the divisibility by 7.

Repeatedly speaking if we compose the number writing down together as fragments the numbers each of which is divisible by 7 we will again have that the composed ”big” number is divisible by 7 too.

So if we’ll join together 200 fragments of 28, we will get the 400-digit number

$$282828\ldots28$$

with two hundred 28’s (by the way, who is the world would be able to read it?) and with the sum of digits being exactly 2000.

Now we need to join to this huge number one fragment, which as a number is divisible by 7 and have the sum of digits equal to the modest number 7. Examining our initial multiplies of 7 once again we see that we can take the number 7 itself (if we are not fond to join 1-digital number 7 we can take the number 70 or even 133).

In the last case we could construct the numbers

$2828\ldots.28133$, $1332828\ldots28$ or even

$281332828\ldots28$, $28281332828\ldots28$ etc.

Each of them has exactly 403 digits and sum of digits 2007.

Let us note that we are free to insert the fragment 133 in any place after arbitrary number of 28’s but of course not in the middle of it like

$$2133828\ldots28$$

because then we can loose the divisibility of 7 that we are so eager to preserve.

For example inserting 28 in the middle of fragment 28 we would get the number 2288 which is no more divisible by 7 because

$$2288=7\cdot326+6.$$

MULTIPLIES OF SEVEN: TWO MAIN QUESTIONS LEFT

The first of these general questions asks what is the least possible sum of digits of multiply of 7?

It would be nice to find the multiply of 7 with the sum of digits equal 1. Putting together 102, then 999 or afterwards k such numbers we would get the multiplier of 7 having respectively the sum of digits equal 102, 999, k

That would mean then that each positive integer is the sum of digits of a multiply of 7.

But this is not the case alone from the fact that the sum of digits equal to 1 has the numbers

1, 10, 100, 1000, 10000

or, simply speaking, the numbers which have the first digit 1 with possibly some zero's after it ("one with many zero's"). But each such a number having the first digit 1 with many zero's after it is divisible only by 2's, 5's, their powers and products.

Our next hope - after we've clearly stated that no positive integer with the sum of digits 1 is divisible by 7 - is to find a multiply of seven with the sum of digits 2. Then we of course would start joining them and would get that any even positive integer is a sum of digits of some multiply of 7 and so on.

But can we find such a multiplier of seven with sum of digits 2? Is it possible? What to do? These questions arise always by solving the problems or more generally in everyday life when we start arranging something not very trivial.

How to proceed with now?

One possibility would be go on with writing down further multiplies of 7 or continue the procedure with which we already started hoping the very soon we will find such a multiply of seven.

We would see then the numbers 105, 112, 119, 126, 133, 140, 147, 154, 161, 168, 175, 182, 189, 196, 203 with sum of digits being at least 4 in the case of 112. Still we can get the sum of digits 3 in the case of 21 but this is not 2.

What could be told about the (decimal) expressions of integers the sum of digits of which is 2?

In this case there are two essential possibilities:

(A) There is one digit that is 2 and other following digits are zero’s or

(B) There are two 1’s in that expression.

In the case (A) just as in the case of 1 with many zeros we state that such a numbers are divisible only by 2 or 5 as well as by their natural powers and products and by nothing more.

In the case (B) in the decimal expression of the number there are two digits equal 1 with the possible zeros between them. The zeros after the second digit 1 has no influence to the divisibility by 7 so we may imagine that the expression of this number has digit 1 as its first and last digit or looks like

1000… 0001

So we shall try to do a long division of a number where some zeros follow 1 and patiently waiting for the moment when the partial rest will be 2. Then instead of the following 0 we will take the (second) 1 and the long division will be completed.

In that case the partial rest equal 2 appears almost at the beginning of long division:

1 0 .. . 1 | 7
7   1
3 0

31 is not divisible by 7
so we take one more 0

1 0 0 .. . 1 | 7
7   1 4
3 0
2 8
2

1 0 0 1 | 7
7   143
3 0
2 8
2 1
2 1
0

21 is divisible by 7 so we
already take 1 instead of usual 0

So the sum of digits of a multiply of 7 can be equal 2 and one such multiply we've just found – it's a number 1001. If we were writing them down carefully from the very beginning it would be exactly the 143rd multiply of 7 because

$$1001=7\cdot143.$$

Now all that remains us to do it establish what numbers can be the sums of digits of multiplies of 7?

Putting together two 1001 we get the number 10011001 which remains a multiply of 7 with the sum of digits 4, if we put together three such copies we'll get a divisible by 7 number 100110011001 with the sum of digits 6 and so on: putting together n such a copies we'll get the number 10011001…1001 which is divisible by 7 with the sum of digits 2n.

So we established that every even number is sum of digits of multiply of 7.

We already established that with the odd numbers we'd have slightly different situation because the sum of divisible by 7 numbers will never be 1. And what about other odd integers?

The detail that now it's enough to remember is that the clear multiple of 7 is 21 having the sum of digits only 3.

Now putting together 21 with the fragment 1001 we get the composite number 100121 that is divisible by 7 with sum of digits 5.

Similarly putting 21 together with 2, 3,…, n, …. fragments we get respectively the numbers

1001100121, 10011001100121, 10011001100121

having 10,14,…, 4n + 2 digits and the sum of digits 7, 9,…, 2n+3, …

So the global answer to our problem sounds that every positive integer except 1 is the sum of digits of some integer which divisible by 7.

SOME PHILOSOPHICAL REFLECTIONS AFTER DIVISIBILITY OF 7 PROBLEMS

Solving this problem we behaved exactly as if we we’ve creating any scientific discipline. Namely firstly we gathered some concrete facts and observations (in our case it was the observations what a sums of digits have the first multiplies of 7). Further on from this using some ideas or constructions (in our case it was the idea of putting the numbers together and long division) some intermediate results occurred and conclusions followed (in our case it was clearing that each positive integer except 1 is the sum of digits of a multiply of 7).

Let us state it in slightly more general form:

If the number ABC…Z is divisible by some integer *m* then the integer

ABC…ZABC…Z……………………..ABC…Z

is divisible by *m* as well.

And what’s then after creating such (micro)theories or what’s after questions like these are answered?

Further it always turns out that such (micro)theories or answered questions always rise many new problems - often more that we at the beginning expected.

Mathematicians and other scientists are joking that every solved problem induces more than enough or at least 10 new problems.

Return to our case. Can such a modest and accessible problem create a new problems and questions?

The answer is sure yes, it can. We'll mention some of such possible questions.

1. By what numbers could be replaced the number 7 in order that the answer would remain the same?

2. Describe other possible sets, which coincide, with the set of all possible sums of integers of all possible multiply of some integer m.

Other similar question could be formulated as well.

As an example we can cite a problem from the 2nd Lithuanian Olympiad for youngsters, 2000:

What numbers can be the sums of integers of multiply of 23?

What would be the answer if we replaced the number 23 by 99? Or by 5? Or by 101?

CHAPTER II.
SLIGHT OSCILLATIONS OR ABOUT MICROMOVEMENTS

There are a lot of matters who we regard to be almost of no importance. They seem to be some kind of unnoticeable details or one of many thousands circumstances.

At this place we may remind that if you are climbing into the mountains then every bulge may be of great help – using it you may continue your march and finally achieve the top of the mountain.

Otherwise you can be forced to stop your march.

Let us regard be problem which were proposed for the 7th grade in Minsk city Olympiad A.D. 2004.

We have 5 positive integers. It is known that if we will add any three of them in every possible way we would get 7 different sums and if we will add any four of these 5 integers again in every possible way we would get 5 different sums. We are expected to prove that the sum of all these 5 integers is divisible by 5.

Let's try to do something in the direction.

Let's give standard names A, B, C, D, E to these five. Then all possible sums of 4 integers are

$$A+B+C+D,\ A+B+C+E,\ A+B+D+E,\ A+C+D+E,\ B+C+D+E.$$

We remind the words of problem saying us, that all these 5 sums are different.

So the 1st conclusion would be that then also all these given integers are different, otherwise some two sums of 4 summands would be the same.

We can also order these 5 integers by magnitude and assume that

$$A < B < C < D < E.$$

The 2nd conclusion could be that if adding in every possible way 3 of 5 of these integers we get 7 different sums then also adding them in pairs in every possible way we also would get 7 different sums.

But in general adding 5 different integers A, B, C, D, E in pairs we may get 10 sums:

$$A+B,\ A+C,\ A+D,\ A+E,\ B+C,\ B+D,\ B+E,$$
$$C+D,\ C+E,\ D+E.$$

Because in our concrete case there are promised only seven different sums this indicates that some of these sums coincide.

Now we are trying to find such sums of 2 summands that are always different if initial numbers were different.

It is clear that

$$A+B < A+C < A+D < A+E < B+E < C+E < D+E.$$

That we have only 7 different sums adding in pairs means that the remaining 3 sums B+C, B+D, C+D must coincide with some of 7 already mentioned different sums. These remaining three sums are clearly ordered by the magnitude, namely it is as easy as possible to mention that

$$B+C < B+D < C+D.$$

With what of these 7 sums do coincide e.g. B+C? It is greater then the second sum A+C that's why it could coincide with the third sum A+D or the fourth sum A+E (the fifth sum B+E is already greater than B+C).

Similarly from the other side C+D is less than C+E but greater than A+D that is it coincides either with the sum A+E or B+E.

Finally B+D is greater than A+D but less than B+E that is it must be equal to A+E.

A+B	A+C	A+D	A+E	B+E	C+E	D+E
		B+C	B+D	C+B		

That is B+C is the third, B+D the fourth and C+D – the fifth of these seven sums, that is

$$B+C=A+D,\ \ B+D=A+E\ \text{ and }\ C+D=B+E.$$

Rewriting it in a slightly different way as

$$B–A=D–C,\ \ B–A=E–D,\ \ E–D=C–B$$

we see that

$$B–A = C–B = D–C = E–D.$$

This means that all differences between any two neighbouring numbers are the same or otherwise that have to deal with an arithmetical progression.

Remark. In this place there's no need to know a word about arithmetical progressions or these "monotonically" increasing sequences of numbers.

Namely from the equality

$$D–C = C–B$$

it follows that

$$B+D = 2C$$

and correspondingly from

$$B–A = E–D$$

we get

$$A+E = B+D = 2C$$

that is

$$A+B+C+D+E = (A+E)+(B+D)+C = 5C$$

So the sum of all 5 initial integers is indeed divisible by 5.

For those who would be eager to repeat something similar to what we just did we would propose to prove the problem, which was suggested, in the same Minsk Olympiad for the grade 8.

It is possible to represent the number 2004 as a sum of different summands so that adding these summands in pairs we get exactly 7 different sums?

In highest grades nicer modifications of that idea were to be seen:

We are given 100 different real numbers. It is known that the least of them is 0.08 and the greatest one is 40. Also it is known that adding them in pairs in

every possible way we get 197 different sums. Find the sum of these 100 numbers.

And another question which is no more quite **trivial**:

Find the least possible and the greatest possible positive integers n such that it would be possible to find n positive different numbers such adding them in every possible way in pairs we get 2004 different sums.

We could propose several other similar problems as well. For these who are not yet tired or completely bored we propose to read the following chapter.

CHAPTER III.
ABOUT ONE STAMP COLLECTION

The son of our neighbouring professor that name is Mr. Meridian a month ago started to gather stamps. William – so was his name – was always amazed by the Scandinavian countries so that no wonder that he is gathering the stamps of Denmark, Sweden, Norway, Finland and Island which for him is a hundred per cent Scandinavian country.

While he just started so he has only a few stamps of each country. In order that it would seem more once he counted the stamps of each possible pair of countries and has written down very carefully all sums he's got.

Examining these numbers he was deeply surprised by the fact that he's got only three different sums. He counted once again – again the same but only three different sums appeared **– 13, 18 and 23.**

So his consulted his father who was always very fond of any kind of puzzles especially with mathematical flavour and always knew how to understand and explain them.

Father listened to his sun very carefully and after some minutes of reflection informed the sun that in this case everything is O.K with his counting. Moreover the declared that if the sun is not able looking to the numbers to tell how many stamps of every country he actually has, then he, professor, could help to calculate it.

He expected rather that his sun would not allow he to do this because he would like to find that out himself. His was right in his expectations because William told exactly that he would like to do it by his own.

Is it possible having only 3 sums to find out all 5 numbers?

The problem is simple and in the same time rather interesting. It is based on problem Nr. 259 from the Minsk Olympiad book of problems for grades 5 till 7. (see [2]).

After a half an hour of deep thoughts our hero was able to understand that the quantity of stamps of each Scandinavian country can’t be different because in such a case he would have at least 7 different sums of pairs (just as it was told in the previous chapter) and not 3 as in his case.

Remark. The reader understands pretty well that William could count the number of stamps simply opening each a collection but we must know that he always used every possibility to employ and develop all his thinking powers and capacities.

After another half an hour William understood that it will be more than 2 countries with the same number of stamps in his collection.

Having only one pair of countries with the same quantity of stamps we would have some 4 countries with different number, say A, B, C and D, of stamps. Without a loss of generality we may assume that these numbers are ordered by the magnitude.

Then similar as in the previous chapter we could state that

$$A+B < A+C < A+D < B+D < C+D.$$

So we have already 5 different sums instead of 3 as it is in William’s case.

After that he regarded the possibility that there are two pairs of states with same number of stamps and noticed that this is impossible because then adding them in pairs he would immediately get two even sums – and he’s got only one.

In similar way William eliminated the case when there are only two states with the different number of stamps. If 4 states have the same number of stamps different from the fifth, when we won’t get 3 different sums but just 2. Other possibility is for 3 states have same number of stamps and for the remaining two also the same but different number of stamps. Then counting in pairs we again would get two even sums – and we have the only even sum 18.

So it remains the case that the three states have same number of stamps different from the fourth and also different from the fifth state.

In that case adding the stamps of these 3 equal-stamps states in pairs we will get the even number that is 18 so it follows that William has

$$18 : 2 = 9$$

stamps of some three Scandinavian states. So the number of stamps of the fourth state is

$$13 - 9 = 4$$

and

$$23 - 9 = 14$$

is the number of stamps of the fifth state.

We are very fond that William mastered his problem and that we were able to understand how was he thinking and what was he doing.

CHAPTER IV.
WHERE DO THE BOUNDARIES OF OUR POSSIBILITIES LAY?

What are my possibilities? Are they really so great or at least remarkable? Where lay their boundaries? Is it possible to increase them? What ought I do in such a case? What is possible to achieve in one or another situation and what is not realizable and why? How to distinguish these two cardinal cases? Distinguishing how to prove it?

These simple eternal questions excited human beings as Homo sapiens from the very first day Homo sapiens started to think and wonder. They excited the fantasy and inspired him to think and act, to look and try again.

Sometimes, especially when no progress is to be seen, the wish drop all this or at least lay aside and forget it may appear.

All that is understandable, normal and human. In such a case you could simply take your time to

recover yourself and by suitable circumstances to return back to these questions and problems.

In mathematics there are almost unbounded possibilities for all this, especially for training of abilities to distinguish what is possible and what is not.

Let us consider a simple possible exercise for developing of fantasy and thinking art.

The problem we going to consider was once proposed in the International Kangaroo competition provided every year on the 3rd Wednesday of March.

Actually more than 3 MIO participants from 3 continents take part in this affair.

What is the largest number of consecutive integers such that sum of digits of every number is not divisible by 5?

Probably at first thing we remember is that every fifth positive integer is divisible by 5. This is indeed so but meanwhile we are speaking not about the divisibility of the integers but rather about the divisibility of their sum of digits: in this case it is no more right that every fifth number has sum of digits divisible by 5.

Let us again make a concrete experiment regarding e.g. some 2- digital numbers in order to get some idea what may then happen.

For the technical convenience let us formulate our task in the following way.

How many consecutive positive integers with a sum of digits indivisible by 5 could be found between two positive integers with sum of digits divisible by 5?

From now on in these chapter integers with the sum of digits divisible by 5 we will write bold.

If we take **23**, 24, 25, 26, 27 and **28** then we have the situation with four consecutive integers with sum of digits not divisible by 5 between two integers **23 and 28** with sums of digits divisible by 5.

But as it was told it is not always the case. We can have not 4 but less of such consecutive integers: e.g. **87**, 88, 89, 90, **91**.

In this case we have only 3 such consecutive integers.

It is possible also to find more than 4 of such consecutive integers e.g. 7 as in case below

96, 97, 98, 99, 100, 101, 102, 103, **104**.

So once again we repeat our question: what is the greatest possible number of such consecutive integers?

The answer could be given after one almost obvious observation.

If the have consecutive integers with no shift in the unit digit then the sum of digits of these integers is always increasing by 1. So in that case when there is no shift in the unit's digit it is possible to find at most 4 consecutive integers with sum of digits indivisible by 5, for example

14, 15, 16, 17, 18, **19**.

So that the idea is simple: find 4 consecutive positive integers with sum of digits indivisible by 5 and with no shift in unit's digit, then a shift from the units digit to the the digit of tenths follows and then again another 4 consecutive integers with indivisible by 5 sums of integers and with no shift in the units digit may follow..

That means we can have at most 8 consecutive integers with indivisible by 5 sums of digits.

Clearly we need an example for it and such example lies at hand:

6, 7, 8, 9, 10, 11, 12, 13.

Let us add the some related questions.

1. "Symmetrically" it could be also asked: what is the least possible number of integers with indivisible by 5 sum of integers between two integers having divisible by 5 sum of integers?

2. What is an answer to that question if we would replace the number 5 by the number 6, that is what would be the greatest possible number of integers with the sum of digits indivisible by 6 between two integers having divisible by 6 sum of digits?

3. And what if we'd take 9 instead of 6?

4. And what if 11?

It lies already quite close to the question proposed in the International team-contest "Baltic Way".

5. Find the greatest number of consecutive positive integers with sum of digits indivisible by 13.

We would like loyally inform the reader that the answer to the third question with 9 is similar to these with 5 and 6 but the case with 11 and 13 may prepare us some surprise alone from the reason that the greatest number of consecutive integers with no shift in units digit is 10 (no wonder, we are using the decimal system!).

CHAPTER V.
HOW TO LESSEN THE IMPRESSION OF LARGE NUMBERS?

If you would ask anyone a question: "Are you afraid of large numbers?", then you of course would hear either the answer: ”Why should I?” or “No, never” meaning practically the same.

Then we could rephrase the question and meeting a cowboy to make some inquires what flock is easier to pasture – that with 5 quiet cows or another with 50 wild oxen. Of course even now the enthusiastically sounding answers of the kind “the more wild animals I have to pasture the more challenging it is for me” or something of the kind would be often repeated.

But in fact coming nearer to a raging herd we could change our opinion imperceptibly to an opposite one.

Of course the greater the challenge the bigger the experience and more interesting our tales after some years after survival but otherwise it’s quite clear that if you are not used to take care about dozen quietest chicken then you won’t be master dealing with herds of relatively wild horses.

Or simply speaking – everything begins from smallest things.

But some paradox of the kind expressed by words

A child is a father of the man

always remains.

In order slightly amuse the reader we would like to ask the riddle:

What is the longest English word?

The answer can slightly shock everyone who expects and looks for some really long words in the sense Irish language have.

The answer that I’ve found many years ago in some English book made me sure that the longest English word is – and in the last second I’ve decide to tell it a bit later.

Now I am only going to tell you that the longest English word has 6 (in words: six) letters.

I honestly confess that at that place I would expect from you the natural question or even a scream:

- *Why it's so? This seemingly longest English word according to your claim counts only 6 letters. For example word* **BEAUTIFUL** *counts 9 letters that is one and a half times more. Not to speak about the word* **INFLAMMATORY** *which counts 12 letters so it is already twice as long as our shocking word* **SMILES.**

But all this is only the a start. We can find further rather effective word **INEFFECTIVENESS** *counting already 15 letters and being already two and a half times as long as a word announced by you.. There is no doubt that many longer words could be found.*

This would be a normal standard view for finding an answer. After hearing the answer we've promised and are going to present the reader would be able to enjoy another possible view to the length of words.

The longest English word is SMILES because it's a MILE between his first and last letter.

You accept it when you hear it but to find it using your own mind only wouldn't be easy.

Maybe do you want to guess another riddle?

Which two letters of English alphabet have eyes?

Let us finish this literary digression with the limerick with some arithmetic flavour and content:

There was a young lady for Lankashire
Who once went to work as a bank cashier,
But she scarcely knew
1 + 1 = 2
So they had to revert to a man cashier.

Let us return to the reality of numbers or to the question how to reduce the possible huge numbers when solving the problem and in the same time to lose intrigue and content of the task.

We are eager to simplify the situation to such which is as clear as it is possible (and then we'll be able to return successfully from a couple of chickens to thousands of wild horses).

Let us regard the following problem, which is formulated in the usual so called general form.

Two positive integers n **and** k (probably enormous large) **are given. The question is whether it is possible to find the third positive integer** a **such that the sum of digits of any number**

$$a, 2a, 3a, \ldots, (n\text{-}1)a \textbf{ and } na$$

is divisible by k?

Firstly we should try to make oneself at home with the situation created by the cited problem in order to understand where the possible difficulties of the task may lay.

In this case it is possible to simplify the situation not losing the intrigue and content of the problem.

Let us take instead of an abstract and probably huge number n take a concrete small and very familiar number 10 and instead of k another even more familiar number 2 and ask whether it is possible to find the third positive integer a such that the sums of digits of all 10 numbers

$$a, 2a, 3a, 4a, 5a, 6a, 7a, 8a, 9a \text{ and } 10a$$

are divisible by 2 or, simply speaking, are even.

In this simplest situation it is difficult not to find such number a because taking already of it 11 we see that the sum of digits of all 10 requested numbers

11, 22, 33, 44, 55, 66, 77, 88, 99 and 110

are really even.

If we would wish to possess $n=20$ such integers than continuing the series of multiplies of 11 we would get consequently

121, 132, 143, 154, 165, 176, 187, 198

and now we lack only 2 numbers till the wanted 20 – we have already 18 of them – but on the next step we are forced to take the next multiply of 11 or the number 209 which the sum of digits unfortunately being $2+0+9=11$ or odd.

We failed when only two steps till our aim were left.

Otherwise it's no wonder that when n increases we have more troubles.

After some dawdling we find the number $a=101$ the first multiplies of which or numbers in series

101, 202, 303, 404, 505, 606, 707, 808, 909

remains even much longer or at least until number $101 \cdot 99 = 9999$. That is not the end of that happy fairy tale because next multiplies of 101 are

10 100, 10 201, 10 302, …, 10 908

still possessing an even sum of digits.

But the next multiply of 101 is 11 009 with already odd sum of digits. So we established that the first 108 multiplies of 101 possess an even sums of digits and the 109^{th} multiplier of 101 already an odd one.

And what if we were asked to find 1000 and not 100 such integers? Dear reader, have you already the feeling what to do and how to proceed?

Until we needed 100 multiplies of some integer with the even sums of digits then we've used the number 101. In this number two 1's is separated by one 0. If necessary these two 1's could be separated with much more 0's between them.

After we've understood what to do in a case $n = 1000$, we know how to behave in the case $n = 1\,000\,000$ or with any other number – we only will need perhaps more and more 0's between these two bordering 1's.

So we are able to find the series of integers of any length

$$a,\ 2a,\ 3a,\ \ldots,\ \mathrm{n}a,$$

having even sum of digits of any member of it.

Now it remains to make only one step in order to find the series of any length of multiplies of some integer with the sum of digits divisible not by 2, but already by 3. Everything would appear so simple if we could come across an idea to separate by zero's not two but more 1's. That will work and we'll be done.

So the well known truth may be confirmed: if you are able to understand what to do then everything seems to be so clear that you can't drop away an the thought: if this is understandable for me then it would be also understandable for everybody who will listen to it.

It's quite a nice feeling.

CHAPTER VI. 2009 1'S AND 0'S JUMPING AROUND THE WHEEL

We are already acquainted with the clever teenager William Meridian, whose father is professor for geography (and not only). William is always in motion and can be seen and met everywhere. From the point of view of his everywhere lasting presence it's no wonder that he once found a big wheel. This wheel has exactly 2009 empty entries and it was clearly indicated that in each entry it's possible to put in either 0 or 1. That inscription was located in the centre of wheel in 10 world languages including Latvian, Icelandic and Lithuanian language. In the end of instruction it was also mentioned that putting another integers would destroy the wheel.

William with most possible care took that wheel home and immediately put in its entries 2009 numbers – 0's and 1's. In what way he did it he can't now remember but he was more that sure that not all numbers he's chosen were equal.

Tired after all these efforts William fell asleep and in dream saw the best friend of their family professor Longitude accompanied by his assistant Mr. Wise. William was asked whether he's aware what a wheel had fallen into his hand? Mr. Wise added this is a wheel of wisdom. William asked at once why?

Mrs. Wise advised him to listen carefully and asked:

- Have you ever seen a whistle?

- Of course, - answered William.

- Take into account, - said professor Longitude who was the magician of numbers handing William this whistle, - and don't forget that every time you'll whistle, the numbers put in the entries in your wheel will change in the fundamentally way.

- What is that? - Murmured boy.

- Fundamentally means, - Mr. Wise pronounced all his words very clearly and slowly, - that if two neighbouring numbers in this wheel are equal then between them a 0 will shoot up and if two neighbouring numbers are different then between them a 1 will appear. But most important thing is that *after all these changes the old numbers will vanish away and would be replaced in the same order by the new ones.*

- And what if I'll do another whistle, will I then see similar changes, - asked William.

- Exactly. Again between any equal neighbouring numbers a 0 and between any different neighbouring numbers a 1 will shoot up and *then an old numbers will again vanish away and these new numbers in the same order will replace them.*

- For how long will this whistle belong to me?

- Till the very moment when all numbers in wheel became equal. Then your whistle will disappear and the wheel goes to pieces.

After these words both his guests vanished from sight and William's dream.

When William woke up early in the morning his first thought was that all this was only a dream. The wheel stood at his bed just in the same position as in the evening before.

But the whistle on his pillow witnessed that everything what happened in a dream was only a dream.

William immediately whistled and the numbers changed exactly so as he'd heard in dream. He'd whistled again and again for several times – and always with same effect. Suddenly he stopped whistling – he remembered that if the numbers would become equal we would lose his wonder whistle.

William had never forgot that before he started whistling not all numbers between his 0's and 1's were equal. Now after he'd stopped whistling he carefully examined all entries of wheel and with relief stated that again both kinds of integers were present. He became very curious whether the following thing could be surely established.

Knowing only that the wheel has 2009 entries with either 0 or 1 in each of these entries and that not all numbers are equal is it possible to make oneself sure that it can't happen that after some number of whistles all the entries will became equal?

This would mean that wheel would be broken down.

Trying to grade his task William sought for answers to the following questions:

Won't the whistle vanish away and wheel go to pieces after:

(A) 100 whistles;

(B) 1000 whistles;

(C) 2007, 2008 or 2009 whistles;

(D) After such a great number of whistles that William will became completely exhausted again or in other words after n whistles doesn't the whistle disappear?

What the reader is now being asked for now is some arrangement of a problem which was once proposed on the Lithuanian Team-contest in mathematics for high-school students.

How to master it? 2009 entries are not 9 entries. What to do? That 2009 is not 9, that's clear. Let us take even the number of entries, which is less even than 9, say, take 4 and 5 entries only. Perhaps everything then will became so clear that afterwards we would be able to master the case with 2009 entries.

So let’s start with tiny wheel having only 4 entries and being in such a state as it’s indicated below:

0 1
1 1

After first whistle these numbers would be replaced by the following collection of 0’s and 1’s:

1 0
1 0

This in turn would be replaced by

1 0
0 1

Now after the third whistle we would have

1 1
1 1

or that in the next moment William is without the whistle and remains only with the broken down wheel. So the situation with a tiny wheel with 4 entries is not stable – is it so because 4 is an even number? Perhaps. Who knows? We must look.

Anyway, we will try to imagine the situation with another tiny wheel possessing 5 entries. 5 is odd number. Will it change the situation? Let’s see. In order to get some impressions what the things look like we take again some collection of 0’s and 1’s and see what could happen after some whistles:

1 1	0 1	1 0	1 1
1 1 0	0 0 1	0 0 1	1 0 1

We see that if we turn the first collection of 0’s and 1’s we would get the last one – both of the have four consecutive 1’s and one 0 so that further everything will repeat again.

Now we'll came to the following facts: **independently from the number of entries if in the wheel both numbers 0 as well as 1 are presented then unavoidably some of 0's would be neighbouring some of 1's and consequently after any whistle between these neighbouring different integers a new 1 will appear.**

We would like to repeat this once again: having 0's as well as 1's on the wheel means they avoidably met somewhere as a neighbours and so they'll generate a 1 after a whistle.

This is exactly as in Zoo: if we have elephants and kangaroos standing around the circle then there exist such elephant that is the neighbour of some kangaroo.

If there are at least two elephants and at least two kangaroos staying around the circle then we can state slightly more - that there also another kangaroo having as a neighbour another elephant.

Now what circumstances guarantees us survival of zeros?

In order to guarantee survival of 0's we must guarantee the neighbourhood of two equal integers. For that it is enough the avoid the worst situation when 0's and 1's are placed in every second place or so that around the circle any 0 is always followed by 1 and any 1 is always followed by 0. This is impossible when the number of entries around the circle is odd and is always possible when the number of entries is even.

So if the number of entries is odd – as in our initial case with 2009 entries – then it is impossible for each 0's and 1's take "every second place" around the

circle. It means that either two 0's or two 1's will be neighbours and after a whistle 0's will also survive.

Under these circumstances it is quite clear that in case of odd number of entries around a circle either 0's or 1's will be present after a whistle if they were present before. From this we conclude also that they will be also present after any number of whistles. It could be shortly expressed that

In a case of odd number of entries 0's and 1's will always be present if they were once present.

And so William with his wheel with 2009 entries will never lose his whistle.

CHAPTER VII.
THE BINARY NUMERATION SYSTEM

After William's adventures with 0's and 1's it would be suitable to discuss the binary numeration system alone from the reason that in this system there no are other digits – just as it was on the wheel nothing but

0 and 1.

More sensitive person could understand a binary numeration system as a world created exceptionally by 0's and 1's.

In our computer times and epoch of huge numbers we all possess some often rather remarkable experience on that field. Everyone surely knows that our usual decimal digit 0 is written also as 0 in the binary system too. Similarly our usual decimal digit 1 is 1 in binary numeration also. But our decimal digit 2 in such a form as it is in written in the decimal system isn't written as 2 in the binary system. For the representing of the number, which in decimal system is written as 2 in the binary numeration system two digits are already being used. So how do we write or how do we express 2 in the binary system or in other words what's the binary representation of 2?

The answer is: the binary representation of 2 is 10, we sometimes even write especially feeling that no confusion may happen that

$$2 = 10.$$

It is quite clear that if 2 in a binary numeration system is expressed as 10 then 3 as 11 and the binary representation for 4 is 100.

We see pretty well that the length of a binary representations comparing them with the decimal ones "grows" rapidly and e.g. our usual 8 in binary representation is already 1000, 15 – 1111 and the binary representation of 16 or 10 000 employ already 5 digits.

Despite of that the binary system thanks the computers is everywhere used and known it's claimed that when taking into account only the length of representation not the binary but the ternary system of numeration were the most convenient one.

By the way the ternary representation of the first positive integers is

0, 1, 2, 10, 11, 12, 20, 21, 22, 100, 101, …

The last written number 101 in ternary representation is our usual decimal 10.

The arithmetical operations or four arithmetical rules in binary numeration system are carried our similarly as in decimal system. Only it is convenient to know the binary addition and multiplication table.

The table of binary addition is

+	0	1
0	0	0
1	1	10

and the table of binary multiplication is as follows:

×	0	1
0	0	0
1	0	1

Let’s add in column, say, 1011 and 10101:

$$\begin{array}{r} 1\ 0\ 1\ 0\ 1 \\ +\quad 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 0\ 0 \end{array}$$

To explain what we doing is very simple: in units digit 1 + 1 is already 10, we write 0 and carry our 1 over to next digit of tenths then in the tenths digit 1 + 0 and plus 1 which was carried over gives 10. Again we write 0 and carry over 1 to the next digit of hundreds and so on until we get as a sum the number 1 with five 0’s after it. Writing what we’ve just did in the decimal notation we would have

$$32 = 21 + 11.$$

It’s no wonder because 10 000 is 16 so 10 100 is 10 000 + 100 or 16 + 4 = 20 and 10 101 is 21. If 1 000 is 8, then 1 010 means 10 and 1 011 is our usual decimal 11. That is indeed we made the operation which decimal expression is 21 + 11 = 32.

Similarly the multiplication is carried out – let’s look how we’ll multiply 111 by 101 (or simply looking for our usual 5 times 7):

$$
\begin{array}{cccccc}
 & & & 1 & 1 & 1 \\
 & \times & & 1 & 0 & 1 \\
\hline
 & & & 1 & 1 & 1 \\
 & & 0 & 0 & 0 & \\
 & 1 & 1 & 1 & & \\
\hline
1 & 0 & 0 & 0 & 1 & 1
\end{array}
$$

Now we intend to demonstrate how it would be possible to simplify one threatening looking problem proposed in the British mathematical top struggles. After the reader will see the text of problem he will at once understand why the word “threatening” was employed and in which year it happened.

Find an integer whose binary expression contains 2005 1’s and 2005 0’s and which is divisible by 2005.

We would like to simplify the condition resolutely so we would replace 2005 by 5. Or, in other words, we intend to look for an integer whose binary expression contains 5 1’s as well as 5 0’s and which is divisible by 5.

So that our problem “all turns about 2005” is reformulated as an “all turns” about 5”.

This is almost obvious (and that is good so!) because 5 in binary system is 101 then 5 + 5 or 10 or as a binary operation be carried out as follows:

$$
\begin{array}{cccc}
 & 1 & 0 & 1 \\
+ & 1 & 0 & 1 \\
\hline
1 & 0 & 1 & 0
\end{array}
$$

It’s a pity but the number of 1’s in 1010 remains as it was, we’d be glad to get a sum with, say, 3 1’s because then putting together the number we’d got with initial number 101 (putting together was widely discussed in Chapter 1 when we spoke about multiplies of 7).

Again we would like strongly emphasize that putting together numbers as it was being done in Chapter 1 or simply writing together some numbers and regarding it as one integer we do preserve a divisibility in the sense that if all numbers which we'll write as one integer were divisible by some integer *a* so the composed integer will be divisible by that integer *a* too.

Because as it was mentioned the sum 1010 has again two 1's we add once again binary 101 or decimal 5. Now 1010 + 101 = 1111 (or decimally speaking 15, again not so good 1111 contains four 1's – too much!). Let again add 5, we'll get 20 or in binary form

$$1\,111 + 101 = 10\,100$$

(again only two 1's, we are waiting for 3). Adding yet once again 5 we get 25 or binary speaking 11 001. This divisible by 5 integer with three 1's is the number we're waiting for.

Now putting (or writing) together 101 with 11001 we will get again divisible by 5 integer 10 111 001 which contains already five 1's but still only three 0's. It remain to write at the end of the number two 0's and we would get the number

$$1\,011\,100\,100$$

which is the solution of our problem because it contains 5 0's, 5 1's and is divisible by 5.

Remarks.

1. It was possible to put (write) together the numbers 101 with 11001 in another way composing the number 11 001 101 and of course preserving the divisibility by 5.

2. These two 0's written in the end of the composed number without disturbing a divisibility by

5 could be written in other places without disturbing a divisibility as well – for instance, both 0's could be written between the composed numbers and we would get the number 1 010 011 001 or 1 100 100 101. Also we could write one 0 between the composed numbers and another in end of it getting the numbers 1 010 110 010 or 1 100 101 010. So it is possible to insert 0's everywhere between the composed fragments and it's not possible to insert them in the middle of the fragments of composed numbers.

It would rather interesting to know what a usual decimally written integer represents any of the answers of our problem "all turns around 5", say, the number with binary expression 1 100 110 100. And additionally one rather curious thought might be: is it really divisible by 5?

It is so simply to establish it in a usual way:

1 100 110 100=

=1 000 000 000+100 000 000+100 000+10 000+100,

then writing as a "usual" sum it would be

512 + 256 + 32 + 16 + 4 = 820.

820 ends with 0 and as every such integer whose decimal expression ends with 0 is divisible by 5.

It's worth mentioning that the analogical problem "all binary turns around 4" or how to find the number whose binary expression contains 4 units and 4 ones and which is divisible by 4 could be done in the same way on it would much more easier because the binary expression for 4 is 100 and the usual arithmetical sum 4 + 4 = 8 in the binary world looks like

100 + 100 = 1000.

Further the binary expression for 8 + 4 = 12 is

$$1\ 000 + 100 = 1\ 100$$

and it very convenient for our task because writing together two such numbers whose binary expression contain 2 zeros and 2 ones we'll get a number

$$11001100,$$

containing 4 ones and 4 zeros already and which in the same time divisible by 4. This binary written number 11001100 is decimally speaking and writing 204 because

$$11\ 001\ 100 = 10\ 000\ 000 + 1\ 000\ 000 + 1\ 000 + 100 =$$
$$= 2^7 + 2^6 + 2^3 + 2^2 \text{ or } 128 + 64 + 8 + 4 = 204.$$

An easy exercise for the mind gym suits an analogous problem "all turns around 3" or a wish **to find one such a number whose binary expressions contains 3 zeros and 3 ones and which is divisible by 3.**

The easiest problem of such a kind would be a request **to show us a number whose binary expression contain only 2 zeros and only 2 ones and which is divisible by 2 or is even.**

Curious circumstance in that case is that our usual putting or writing some fragments together as one number doesn't work any more because in the wanted number there ought be "too few" ones.

The binary expression for 3 is 11, so 3 + 3 = 6 in the binary world is written as 11 + 11 = 110. Further on the binary expression for usual

$$6 + 3 = 9 \text{ is } 110 + 10 = 1\ 001.$$

We'll continue adding 3's and waiting for a number with three 3's in its binary representation. Our waiting is not especially long because

1 001 + 11 = 1 100,
1 100 + 11 = 1 111 (four 1's is too much for us!),
1 111 + 11 = 10 010 (again only two 1's),

10 010 + 11 = 10 101 (finally we'd got what we wanted to).

This means that the example of a number divisible by 3 with 3 zeros and with 3 ones is a binary expression is

101 010 (a 3rd zero in the end of the number 10 101 is written).

And in a case of already mentioned easiest problem all we need is to write one line because the binary expression for 2 is 10 so that as a wanted example is the number 1010, which was build up from two "copies" of 10 or our usual 10.

In a quite similar way the reader could successfully master the original British problem with divisibility by 2005 with 2005 zeros and ones. It ought only be honestly added the binary expression for 2005 is much longer and adding 2005 + 2005, 2005 + 2005 + 2005, …, and waiting until (what?) will cause more difficulties but exceptionally of the technical nature..

But there are no other differences as in the cases, which we just regarded - only longer expressions for the numbers in question.

CHAPTER VIII.
DON'T ALWAYS CATCH WHAT'S LYING NEXT TO YOU OR NOT EVERYTHING IS GOLD WHAT LOOKS LIKE

Once in already many times mentioned Kangaroo contest the following problem was proposed:

Regard all integers from 1 till 999 and count the sums of digits of all of them. Afterwards count the sums of digits of the numbers you've got or count the

sums of digits of the sums of digits. What's the greatest number we've got doing this?

It's more than clear that from all integers from 1 till 999 the greatest sum of integers has the greatest of them or 999 and this greatest sum of digits is 9 + 9 + 9 = 27.

At this place it would be quite natural to have an illusion that if the integer 999 has the greatest sum of digits among all integers from 1 till 999 then also 999 will have the greatest sum of digits of its sum of digits. We repeatedly claim that this is an illusion but this illusion appears so natural. But is it really so? Yes, it is.

It's again and again true that between all integers from 1 till 999 the latter one has the greater sum of digits 27 and all these possible sums of digits are all possible integers between1 and 27.

Now from all integers from 1 till 27 it's no more the last integer 27 with the greatest sum of digits equal 2 + 7 = 9 among but exactly 19 which sum of digits 1 + 9 = 10 is clearly greater.

Another funny similar problem on the kind is the one we've found in one of the beautiful books of Sankt-Petersburg math competitions (see [3], problem 95.18):

Determine 6-digital integer divisible by 8 with the greatest sum of digits.

Clearly all 6-digital integers are these from 100 000 till 999 999 and again it would appear so natural to apply to the greatest between them and which is in the same time divisible by 8 or to 999 992. This number 999 992 possess quite remarkable sum of digits

$$9 + 9 + 9 + 9 + 9 + 2 = 9 \times 5 + 2 = 47.$$

But the number which is divisible by 8 and is less or an integer 999 984 = 999 992 – 8 has a sum of digits

$$9 + 9 + 9 + 9 + 8 + 4 = 48,$$

which is greater then that of 999 992.

Moreover, another yet lesser integer 999 976 = = 999 984 - 8 has a sum digits which is yet greater than that of 999 984 because

$$9 + 9 + 9 + 9 + 7 + 6 = 49.$$

We believe that any smart girl or boy in the grade 5 would solving this problem would came at once across to a number 888 888 as to an example of the 6-digital number whose divisibility by 8 is most obvious among all 6-digital integers having quite remarkable sum of digits $48 = 6 \times 8$.

But the sum of digit of number 999 976 is not the greatest one. We claim and will prove that among all 6-digital divisible by 8 integers the greatest sum of digits possess the integer having half of 9's and half of 8's in its decimal expression or the integer

$$999\,888$$

having sum of digits

$$9 + 9 + 9 + 8 + 8 + 8 = 3(9 + 8) = 51.$$

To understand that this is indeed so is possible to reason as follows: taking into account that 1 000 is clearly divisible by 8 because

$$1\,000 = 8 \times 125,$$

and adding 1 000 to the number 888 888, whose divisibility by 8 is the clearest one among all 6-digital integers we get the integer 889 888 = 888 888 + 1 000 which again is divisible by 8 as a sum of two such integers.

But also 10 000 and 100 000 are divisible by 8. Adding them also to 888 888 will give the integer 999 888 surely divisible by 8.

By the way, the criterion of divisibility by 8 may be formulated as follows: the integer is divisible by 8 if and only if the number formed by its three last digits is divisible by 8.

Speaking "more scientifically" positive integer M and the integer R formed by the last three digits of M have the same rest when divided by 8.

This criterion is based upon the fact that 1 000 and all its multiplies or numbers of the form

ABCD…WXYZ000

are all divisible by 8.

So taking any divisible by 8 3-digital numbers and writing "in front of him" any integer we'll have an enlarged integer, which remains divisible by 8 (this we did in fact with integer 888 adding three 9's and getting 999 888).

We will now strictly prove (prove and strictly prove is the same but strictly prove from psychological point of view makes upon a person a greater impression) the no other divisible by 8 6-digital integer can possess the sum of digits greater than 51.

From human point of view almost every claim that you can't do something often makes a remarkable impression. The statement of the kind that this or that is impossible sounds very powerfully.

Our reasoning in this case will in no way be complicated - **it's enough to prove that no 3-digital divisible by 8 integer can possess the sum of digits exceeding that of 888 or 24.**

Let's assume that there is an integer having greater sum of digits greater than 24. Then this sum might be 25, 26 or 27 and nothing more. We intend to regard all these 3 possible cases one after another:

1. No divisible by 8 integer can't have 27 as sum of its digits because the only 3-digital integer with sum of digits 27 is 999 and this number is not even even, that is this integer isn't divisible by 2.
2. No divisible by 8 integer has 26 as a sum of its digits because there are only 3 such integers, namely,

   899, 989 and 998.

   The first two of them aren't even even numbers and the third though already even but even not divisible 4 – and we need more – divisibility by 8.
3. If the sum of digits of 3-digital number is 25, then comparing it with 999 we can state that:

   Either one of its digit is two units less than 9 that is one of its digits is 7 and others 9;

   Or 2 from 3 of its digits are one unit less than 9 that is 2 from 3 of its digits are 8 and the third one 9.

   In the "either" case there are 3 candidates":

   797, 979 and 997

   We conclude this case stating that they all aren't even even integers.

   In the "or" case we are to regard another 3 numbers

   889, 898 and 988.

   All is already done because clearly 889 isn't even, 898 isn't divisible by 4 and finally 988 though divisible by 4 isn't divisible by 8 because its distance from divisible by 8 "round" integer 1 000 is12 and 12 isn't divisible by 8.

Again it could be noticed the computer would smile if only it could when asked to solve such a problem: in a smallest parts of a second it would simply check all divisible by 8 integers indicating wanted greatest sum of digits.

CHAPTER IX. OR TWO WORDS TOWARDS THE IMPORTANCE OF FORMAL THINGS

Sometimes it's not easy to regard some small from the first point of view details with necessary respect because from psychological point of view we are used to make a difference between essential on not essential matters. It's understandable and right but on the other side our own experience learns us that that there are so many relative things in world and in science and surely everywhere.

In one or another way it is always possible to measure all things with a proper measure and take them into account in order to distinguish which details are small and which are huge.

Recall the problem which appears to be classical in the modern elementary (or such which is accessible for everybody who's willing to understand it) and perhaps even belongs to absolute classics of the human thinking.

Imagine that being on the street we invited for a visit first 6 persons we met. Can you ever imagine that independently what persons we invited among then always:

either there is a group of 3 persons such that each of these 3 knows other 2;

or there is a group of 3 persons such that none of these 3 knows other 2?

It ought to be solemnly added that this “either…or” doesn’t exclude the case with successful finding both kind of groups in the same time.

In this place again it would be so important to speak so understandably that it wouldn’t be possible to speak more precisely. That’s what we’d like to achieve.

What to do then? How to be understandable to each who is listening to?

No one can give the definite answer to that question. From the other side there are so many possibilities for achieving it which seem worth trying.

AN ATTEMPT TO EMPLOY WHITE AND BLACK RIBBONS

Again we declare the wish to be maximally understandable. Doing that we intend always to raise the question whether it couldn’t be done more perfectly? Couldn’t it be done more understandably? More precisely?

Let’s start from the indisputable truth that every 2 persons among these 6, which are actually paying us a visit either know each other or do not know each other.

Either they are acquainted or they are not acquainted.

There are no intermediate states like that being “half acquainted”.

Now let’s equip each such a pair with a ribbon.

If this is an acquainted pair let’s deliver them a **white** ribbon. May one of them keep one and another other end of that ribbon.

If this is an unacquainted pair we'll do the same but only with **black** ribbon.

After we're finished with this procedure let's meet them all once again and see what they are keeping in their hands.

We'll see that each of them is keeping 5 ribbons each of these is white or black and which connects each of them with resting 5. And our task now is in almost imperceptible manner transferred into the following one:

Find the completely black or completely white triangle where the vertices of triangle mean persons and sides are ribbons.

How to find such triangle? Assume that we can't find such triangle. From which side would the contradiction arrive?

Let's approach the nearest person. This person as every other of them holds an ends of 5 ribbons. These ribbons can be of one or of different colours.

Assume firstly that all these ribbons are of one colour, say, they are all black.

Then no pair of these 5 persons holding another ends of these 5 black ribbons can be connected between themselves with black ribbon – otherwise taking such 2 persons together with the approached one we would have already a completely black triangle.

If really no pair of these 5 persons holding another ends of that 5 black ribbons can be connected with black ribbon then each possible pair of these 5 persons is connected with white ribbon. Then will would have not just one but even 10 possible white triangle. Here - as we remember - persons correspond to vertices and sides are ribbons. That all could be demonstrated either in the picture or we can simply count all these possible white triangles.

For the sake of shortness let's give to the firstly approached person a name A and for other five - the names B, C, D, E and F. We will count white triangles indicating their vertices. These 10 triangles are:

(B, C, D), (B, C, E), (B, C, F), (B, D, E), (B, D, F),
(B, E, F), (C, E, F), (C, D, F), (C, E, F) and (D, E, F).

So now we are done with the case when the firstly approached person holds all ribbons of one colour.

In a similar way we are going to regard the case when the person we firstly approached holds 5 ends of ribbons and

These 5 ribbons represent different colours. Because there are 5 ribbons and 2 colours then there at least 3 ribbons of one colour and let this colour be black. Any pair of persons B, C and D staying on other ends of these ribbons is also connected between themselves with ribbons of some colour. If some of these ribbon, say, the ribbon connecting B and C is also black, then the triangle with the vertices (A, B, C) is already completely black. If no ribbon connecting persons B, C and D is black, then they all – BC, BD and CD – must be white. This indicated that the triangle BCD is completely white and finishes our task.

Remark. It could be mentioned that, strictly speaking, the case with 5 ribbons of one colour in one hand could be omitted because it's by the following case. Still we think that the discussed situation is worth repeating.

LET’S AGAIN WALK ALONG THE STREET FOR TO MEET AGAIN ANOTHER SIX

6 persons we can meet even in small village. Before meeting and inviting them we could answer the simplest question: how much pairs of acquainted persons it can happen to be in a case of 6? Suppose that these persons are marked exactly as they’ve marked before so that we could simply count the all possible pairs of (possibly even acquainted) persons. These pairs are:

(A, B), (A, C), (A, D), (A, E), (A, F), (B, C), (B, D), (B, E), (B, F), (C, D), (C, E), (C, F), (D, E), (D, F) and (E, F).

So there could be at most 15 acquainted pairs. It happens when all persons know each other – and such a case in a village is very probable.

Being in a town or in a city very probably we will have also some unacquainted pairs between any 6 and any other number of persons we’d select. It can also happen that there are no acquainted pairs between them.

In general a number of acquainted pairs differ between

0 and 15.

POSSIBLY IMPORTANT REMARK OF PSYCHOLOGICAL NATURE

From the psychological point of view it is absolutely understandable that we, human beings, have the right and privilege under all circumstances try to preserve our personal dignity. We expect also that our views would be honored or at least taken into account. The most terrible possible human complaint could sound as follows: his attitude towards my person is as if I were an invaluable thing which could be thrown away.

We will return also to that problem and now we would like to add one phrase which is connected with the matters we've just touched. This phrase belongs to the famous Polish mathematician Hugo Steinhaus:

Mr. A can't bear any math's book after he'd found in an algebra textbook an equality

$$A = A.$$

Seeking for the possible explanation of such a sad event we could image that probably Mr. A become offended because he could imagine the A on one side of the equality A = A might mean him and another A on another side of that equality is only a letter and such his comparison with a though capital but only a letter was something that he couldn't endure.

There isn't any doubt that even worst human being (we wish for the sake of God that it wouldn't any human being of the kind) is more important that first letters of all world's alphabets together. In the same time in Math – and that is what we are in no way intending to keep back – it so important to simplify the matters maximally and present all essential circumstances in a most obvious way. For the sake of this we'll without any hesitation if necessary resolutely replace persons by letters, leaders by sides and astronauts by points and so on.

An opinion could be expressed that this wouldn't never surprise tourists, pilots and other persons who are used to deal with longer distances and are fully aware of the fact that the more remote you are the more seems your resemblance to a point to be.

All that was just told may possess connection or be related to possible interpretation of problems we are dealing with.

From the point of view of the set theory there is not a slightest difference between, say, 6 persons and 6 different points because it is possible to establish one-to-one correspondence between them so that for each person there corresponds one point or, vice versa, to each point corresponds a person. We could also speak about persons marked with points.

Such one-to-one correspondence we could also interpret as a “correct matchmaking” when the elements of one set X are “paired” with an elements of other set Y in such a way that distinct elements of X are “paired” with the distinct elements of Y and every element of Y is paired with some element of X.

We could make more reasoning on the subject but let’s return to our 6 persons and their corresponding 6 points.

In this case it is possible not only to establish a correspondence between persons and points but even achieve slightly more: to represent also their acquaintances. If a pair of persons is an acquainted pair then interpreting points as persons we can connect such two points with a line segment and if this is not the case then the corresponding two points remain unconnected. Let us note that in the example which was discussed before we used ribbons instead of line segments.

Because in every plane there are more then enough or infinitely many points then it is possible to choose these points corresponding to persons in such a way that line segments which connect them wouldn’t overlap (but they can intersect having one common point). For this purpose it's enough to select the corresponding points in such a way that no three points belong to the same line.

6 PERSONS WITH 7 PAIRS OF ACQUAINTED PAIRS OR 6 POINTS WITH 7 LINE SEGMENTS CONNECTING THEM

He had bought a large map representing the see
Without the least vestige of land
And the crew were much pleased when they found it to be
A map they could all understand.

(Lewis Carroll, The hunting of the Snark)

If one would ask when is better to speak about persons and their acquaintances rather than about points and segments connecting them then to answer that question in a psychologically right way is not at all difficult.

If we take care of you when we're speaking about any problem then in order to make you a bit interested we must also think about how it would be easier for you to get used to it. For this we always try to (re)formulate the given problem in the possibly clearest, simplest and most impressive way.

Later when we are already involved in solving we are going on trying to convey most precisely key moments of reasoning and solution.

Trying to do it properly we almost imperceptibly turn persons into points, acquaintances – into line segments. When can do plenty things of that nature and this do not cause any irritation. Do not cause any irritation because these efforts help to understand what's played or what's *essential at this moment. From the other side after the successful reasoning when we are already able to understand how the main subjects or solution are running nobody is forbidding us - for the sake of simplicity - to turn back from points and segments to persons and their acquaintances.*

Moreover we feel even obliged to do so because in what language the problem was formulated in that language it'd be suitable also to present a solution.

According to this we'll also formulate a problem in language of persons and their acquaintances but solving that problem for the sake of convenience we'll use points in plane and line segments connecting some of them as a tools of proof.

There is no doubt that an interpretation like that is the kind of Aesop's fable language when speaking about animals (points) one give a moral (logical) lesson.

Firstly we'd like to ask the reader to represent in the corresponding language of points and connecting segments the situation when:

(A) There are 7 pairs of acquainted persons in a group of 6;

(B) Among every 3 persons which you may choose from these 6 there is at least one acquainted pair.

For the reader's convenience we'd already visualized persons as a plane points and are waiting till the reader will connect some of these points with 7 line segments (acquaintances correspond to segments) in such a way that between any 3 point we may choose there will be at least one pair of points connected with a line segment (this corresponds to the initial condition that among every 3 persons we may choose there at least one acquainted pair).

If you’ve already drawn such example then you’d notice that then the following holds:

1. There is such a point between these 6 which is an end point of at least 3 line segments – or in our interpretation there is a person among these 6 which has at least 3 acquaintances in this group.
2. There are 3 line segments making a triangle or by our interpretation there are such 3 persons from these 6 which are mutually acquainted or such that each 2 persons from these 3 is an acquainted pair.

Now let’s ask: is it possibly so only in our example or it would so in every such case?

We intend to prove that if *(A):* in a group of 6 persons there are 7 acquainted pairs and *(B):* among every 3 person we may select there is at least one acquainted pair then surely:

1. We’ll guaranteed find such a person with at least 3 acquainted persons among these 6.

2. We’ll guaranteed find also such 3 persons which are mutually acquainted each with other.

This nice exercise is due to **Romanian problem composer Valentin Vornicu** (see [5, p. 10]).

The idea of solution is well – known: this is so-called **double counting or regarding of problem from two different points of view with comparison of achieved results afterwards.**

Double counting often provides a kind or stereoscopically view and that’s why is so useful.

Our situation meanwhile is quite charming because for that what we promise to prove it’s enough to have less than a half of maximal possible acquainted pairs which is 15 in case when 6 persons are involved. Remind that we need only 7 acquainted pairs and 7 is less than one half of 15. Nevertheless having less than one half of possible acquaintances we are able to prove (we didn’t it as yet!) than there is a person knowing 3 of 5 remaining persons (this is again more than a half) and there’ll be also 3 mutually acquainted or knowing each other persons.

It ought of course to be added that second part of the former statement is in the remarkable degree due to the condition that among every 3 persons there at least one acquainted pair, which is rather strong condition – below we’ll enjoy how it works.

And a proof when you demonstrate it seems so simple – and it is! Such proofs were known for ancient Greeks. Being known for ancient Greeks it carries, by the way, a completely Latin name of “reductio ad absurdum”. This sentence is understandable without any translation because it means more or less reducing to absurd or nonsense or to something that can’t exist.

“Reductio ad absurdum” means that from the condition you’ve assumed reasoning in an absolutely strict and logical way you’d get something what is completely impossible e.g. that yesterday in the noon you were in London and in New –York in the same time – and these both are not two neighbouring villages somewhere in Australia – in that case of neighbouring

London and New-York you could imagine that you're staying with one leg in village of London an with another in the neighbouring village New-York.

It can be compared with a logical explosion which differs so immense from a physical explosion and is something precious that's why worth seeing, wondering and understanding.

We'll arrange this logical collapse in few sentences.

Let's assume that ***(A)*** and ***(B)*** is fulfilled but no one from these 6 persons has 3 acquainted persons among other 5. That means that each of them has at most 2 acquainted persons. Going to every person and carefully counting all acquaintances of all 6 persons we'd have at most 12 acquaintances of all of them. Now for one acquaintance we need 2 acquainted persons exactly as we need two persons shaking hands for one handshaking. So these at most 12 acquainted persons means at most 6 acquainted pairs only – but it was clearly told that there was 7 and not just at most 6 acquainted pairs.

This logical explosion means that our assumption about non-existence of person with at least 3 acquaintances failed. **So there is such a person**. We do not know which person it is, but there is such a person for sure.

Now another part of our statements can be proved in one sentence.

Take this person with (at least) three acquaintances. Among these 3 acquainted persons as among every group of 3 persons there must be at least one acquainted pair giving us 3 persons mutually knowing each other.

ONE CONCLUSION AFTER THE PROBLEM OF VALENTIN VORNICU

Let's recall once again the situation with 6 persons and 7 acquaintances. We've mentioned many times that there can be at most 15 acquainted pairs among these 6. This means that if we 7 seven acquainted pairs then all other possible 15 – 7 = 8 pairs are pairs of persons which do not know each other.

We may assume another interpretation when to a person again a point is corresponding but with a line segment we connect now no more a pairs of acquainted persons but a pairs of persons which do not know each other.

In such an interpretation we would have 6 points and already 8 segments connecting them – we'd like repeatedly to remind that now segments connect persons which **do not know** each other. Just as in a case with 6 points and 7 segments we will have that there is a person knowing at least 3 of remaining 5 or - in other language - there is a point which is connected with at least 3 other points.

Unfortunately now we are no more able to prove that there are also some 3 points which form a triangle or that every two 2 of these 3 points are connected. In the language of persons and unacquainted pairs this means that we are not able to claim that there will be such 3 person with no acquaintance between any 2 of them. We can't prove this because we are no more guaranteed that between any 3 persons there will be at least one pair of unacquainted persons.

We'd highly recommend now to take again 6 points in a plain and connect them with 8 line segments in such a way that there would be possible to show such 3 of them that no 2 of these 3 were connected.

It is possible to have 6 points with even more that 8 segments connecting them in such a way that again it would be possible to find such 3 points that no 2 of these 3 points are connected.

Again we raise the **most natural question**: **at most how many segments may connect 6 points that there is still possible to find such 3 points with no line segment connecting some 2 of them?**

Resuming what was told about 6 persons with 7 acquainted pairs among then we are able to make a slightly more general conclusion:

If in a group of 6 persons we find 7 or 8 acquainted pairs among them and if between any 3 of them there is at least on acquainted pair then among these 6 persons:

- there is a person which **knows** or is acquainted with at least 3 of 5 rested persons;
- there is also a person which **do not know** at least 3 of 5 persons;
- there are such 3 persons that every 2 of these 3 are acquainted or know each other.

CHAPTER X.
BACK TO THE STREET LOOKING FOR PEOPLE

We're just walking along streets and avenues meeting people. During 6-person visit we discussed many matters. We know that if we'd select any 6 persons then:

Either we are able to find such 3 persons among them such that any 2 of these 3 persons are acquainted – a case of so called 3 absolutely acquainted persons

Or we are able to find such 3 persons among them such that no 2 persons of these 3 are acquainted – a case of 3 absolutely unacquainted persons.

We are again walking down the street and again stopped 6 persons. What non-trivial facts could we tell about these 6 persons? What could we foretell? It would make a deep impression upon them and would be useful for us as well.

What things would be interesting also from the psychological point of view? Clearly unexpected things are always of that category.

And dealing concretely we could awake reader's interest with fact that:

Taking any 6 person there are such 2 persons among them possessing an equal number of acquainted persons.

It's of no importance what kind all persons we've met.

How to demonstrate it?

Let us protest against it, let's claim that this is not a case. Let's assume it. Reasoning so how do we get a contradiction? What kind of contradiction it'll be? So again back to **reductio ad absurdum.**

Assume we've found such 6 persons having different numbers of acquainted persons between them.

We ask now: in a case of 6 persons for a chosen person what is a maximal number of acquainted persons. In this group of 6 one can have at most 5 acquaintances (it happens if he knows all other persons. It is also possible to possess less acquainted persons – 4, 3, 2, 1 or even none or 0 acquainted persons.

So there are 6 different possibilities - (from 0 till 5) – for a number of acquainted persons among them for a given person. The number of person being 6 all these possibilities - to possess between 0 and 5 acquainted persons - must be realized. Why this isn't possible? From what side will the contradiction appear?

LOGICAL CONTINUATION OF RIGHT IDEAS

This logical continuation sounds as follows: if among these 6 persons there is a person with 5 acquainted persons then this person knows every other person and then among these 6 person is no more possible to find a person which do not know any other person or has 0 acquainted persons among them.

Exactly in the same way – symmetrically or as if we were looking in the mirror- we could claim then if among these person there is a person with no acquaintances among them then there is no person knowing all other persons.

Let us in the slowest speed repeat what was just being stated:

Let all these persons have different number of acquainted persons, then each of these 6 possibilities will take place or there will be:

- One with no acquainted persons among them;

- Another one with one acquainted person;
- A third one with 2 acquainted persons;
- A fourth having 3 acquainted persons;
- A fifth having 4 acquaintances;
- And finally there is the last or the sixth one with 5 acquaintances.

Stop, as we just stated all these cases at a time are impossible because the person with 5 acquaintances **eliminates** a person with no acquainted persons at all.

So the case with the different numbers of acquaintances of all persons is impossible. This guarantees the existence of 2 persons with the same number acquainted persons (possibly with 0 acquaintances)

A second simple but not at all standard observation could be the following: now we are speaking again about 6 persons alone from numerical solidarity with a previous problem. The number of persons is 6 is not essential here. In this problem 6 could be replaced with any other number of gathered persons.

All arguments and words would remain the same as they were.

For example meeting already 7 or 2007 persons we could and would repeat: if there wouldn't be two persons with the same number of acquainted persons then again we would be able to find one with 6 (or 2006) acquainted persons. It would guarantee us that there is no person knowing nobody and such person should exist if all possibilities take place or every person has a different number of acquainted persons.

CHAPTER XI.
WHO IS ABLE TO PREVENT ME?

“Leave him here to his fate – it is getting so late!”
The Bellman exclaimed in a fright.
“We have lost half the day. Any further delay,
And we shan’t catch a Snark before night!”

This is very actual problem because it is common situation that some people are preventing us and we also are probably preventing not so few persons.

We are always planning, organizing also forecasting many things, we would like to achieve this and that and the persons surrounding us either are helping us or are making obstacles or are completely unaware what we are doing.

Also dealing with problems we often meet the situation *when two persons are acting simultaneously: one is starting, another continuing then again the first is in turn, second is again continuing afterwards and so on. Very often to both of them is clear what they should achieve and very often they are aware of it without any saying.*

Usually separate actions which they do in turn are called movements or simply moves and participating persons are called players. When they are finished with their actions if any of them was able to realize his task which is usually opposite to what the other intends to achieve then that person is announced to be the winner. In any case if necessary the reader may think about two chess-players sitting one in front of the other.

If any of them is able to achieve what he's asked to - independently what the other is doing - then it is often said that this player has a winning strategy.

Psychologically if I have the winning strategy then the fate of my partner is predestinated or fatal and unavoidable. In a case when it is my partner who possesses the winning strategy then my fate is fatal and very often I'm not able to change anything.

But even in such a case when our partner possesses the winning strategy even then we shouldn't lose hope – our partner can make some mistake and sometimes we are able to be saved.

And do not forget that in thinking arts this is only a game which can be very interesting, involving often unpredictable one but it is always only a game and nothing more. Don't let you bother too much when you're playing especially logical or mathematical games.

THE CLASSICAL EXAMPLE WITH 100 CARDS WHICH COULD BE MORE

My dear friend Peter proposed we once the following game to play.

We would take 100 cards in each of which an integer from 1 till 100 are written, different integers in each of the cards. We would lay them down on the table in increasing order. Then we would start the following procedure. Peter would start then it would be my turn and so on. With every move for both of us it is permitted to take from the table no more than 10 cards with greatest integers. It is obligatory with every move to take from the table at least one card.

The winner is the one who takes last cards from the table.

Peter claims that the conditions under which he proposes me to play are very noble ones. Namely he's played this game already for some days and I'm a freshman. In the case he wins I ought to pay him 10 Euro and in case when I'll be the winner he would pay me 100 Euro. Paying so much for me he would challenge me greatly.

He proposes to start in an hour and play 10 times.

What ought I to do?

I have an hour to understand what my perspectives in this play are.

I lied on a bed for a minute of rest and fell asleep. I a dream I've seen the situation from that game with only seven cards left. Because in that situation it was my turn then I won taking all of then away from table. After that I begun to scream from a joy and… woke up.

But these 7 remaining cards from my dream I remembered pretty well. After some sleep my mind was so clear with the 20 minutes till the game left. It was as clear as a day that I'm the winner not only in a case with 7 cards left on the table but also when on the table remains less than 7 (but at least one) or even 8, 9 and 10 cards.

So, anyway, if the numbers of cards remaining on the table differs between 1 and 10 then I'm the champion because I have a right to take them all away from the table.

But if I am on turn and there are 11 numbers left then this is a shivering state: I must take at least one card (that is a condition) away from table; on the other side I've no right take them all because there 11 cards and I have a right in one move to take at most 10 of them away. In any case after my turn a number of cards left on table will be between 1 and 10 and Peter in turn will take them all away winning the game and earning 10 Euro.

So it was clear to me that if it is my turn with the 11 cards lying on the table then Peter will be the winner.

But if it is my turn with 12 remaining cards then I'm the champion again because when moving I'll "**make him 11**" taking the12th card away and similarly I will **"make him11"** if on the table will remain 13, 14, 15, 16, 17, 18, 19, 20 or even 21 cards only then I ought to take away more cards.

Now another shivering number of cards - if I'm in turn – are 22 taking any permitted number of cards I'll "fell in" between 12 and 21 numbers and then Peter will "make 11" to me.

I was only 5 minutes left till a moment we've expected to start that I was able to formulate a law ruling the win in this two-person game: if on a table there are correspondingly

11, 22, 33, 44, 55, 66, 77, 88, 99

cards left then a person which actually is in turn loses. So if there is such situation then I must strongly obey the rule: **the number of cards in one move taken by us both away must be 11.**

So not in vain Peter was speaking about 100 cards on the table and about him moving first. He would then of course take the only card away, 99 others will remain, then he will strictly follow the just formulated rule and I will be lost and he'll be a winner earning in addition 10 Euro with each game.

In the next moment I saw Peter opening the door.

– Let's start or what. You'll make much money.

– Peter, are so naïve assuming that I'm not be able to catch the idea what's being performed on stage? Did you really intend to make money playing with me?

– Not at all, man! I'd like you'd think a bit.

– In that case you achieved it, dear Peter!

CHAPTER XII.
100 CARDS ONCE AGAIN OR ARE WE SMART ENOUGH?

He had forty-two boxes, all carefully packed,
With his name painted clearly on each:
But, since he omitted to mention the fact,
They were all left behind on the beach.

100 cards is a sufficient number in order that some curious or unexpected things would happen. We wish only that we could pay enough attention for what's going on and were ready to learn from all kind of intellectual adventures.

William remembered that he has 100 cards in his pocket with all the numbers from 1 till100 written clearly on each – one number on each card. We imagine that his eyes were closed. Or perhaps his eyes were not closed but only tied up with the handkerchief – who could now state that for sure? Perhaps historians only if they we're willing to.

The matter was the following one. Our neighbor Sixfold became rather curious about the following problem where all cards from William’s pocket were involved:

How many from these 100 cards with the numbers from 1 till 100 ought William to take at random from his pocket that for sure among these taken cards it would be always possible to find such 6 cards with their sum dividing 3?

Formulation of that problem can be found in the problem book of local round of Minsk city Olympiad [6].

We note the for the similar question about how much cards must we take in order to have such **3** cards with their sum dividing 3 we already know the answer from the previous chapters **and this answer is that we must take 5 cards**.

The question when solving is always the same – which detail will now play the most important role?

Because the problem wishes us to add 6 integers so we are forced to take at least 6 cards. But taking only 6 cards can it always be sufficient? Let’s look for an example. As usual we’ll take instead of numbers rather their rests of division by 3. So can 6 cards be too few?

Regard e.g. cards 3, 4, 7, 10, 13 and 16. In the language of rests modulo 3 it would be 0, 1, 1, 1, 1 and once again 1. Summing one 0 with 5 1’s we get 5 and 5 doesn’t divide 3.

So taking 6 cards can’t be enough. What about 7 cards? Perhaps 7 cards would do. Taking at random 7 integers 7, 11, 14, 19, 25, 27 and 35 and going over to their rests modulo 3 – this a scientific name for this operation – or to 1, 2, 2, 1, 1, 0 and 2 we see that it is possible to collect a divisible by 3 number from 6 cards. We take all cards without the sixth one and get

$$1 + 2 + 2 + 1 + 1 + 2 = 9$$

so the initial cards without the sixth one will do.

Moreover our concrete activity with numbers allows us to notice the following: if these 7 taken cards present all possible rests modulo 3 then we are done. Really the sum of all seven rests can have a rest 0, 1 or 2 when divided by 3. Exclude from these 7 numbers the card having exactly this rest and you will have 6 numbers with divisible by 3 sum of them.

So intending to find and example when 7 cards won't do we must avoid each case with all possible rests modulo 3. So there can be only 2 possible rests. Some investigation may lead us to an example 0, 0, 1, 1, 1, 1, 1. It differs for a previous example for 6 cards that the second 0 is adjoined.

So 7 cards may also be not enough.

What's now? Many initiative school girls and boys would without any hesitation go over to the case with 8 cards. We ought to confess that that's what we did also – but all our efforts to find an example that 8 cards wasn't enough failed.

This leads us to the attempt to prove that it is impossible to find such 8 integers with the not divisible by 3 sums of some 6 of them.

How could we prove it? What is the proof?

The proof is reasoning where everything what's written is written right – till the very last letter.

So now we are expected to start an attempt to prove that if we'd take 8 integers then it is always possible to select such 6 of them with a divisible by 3 sum.

The main argument in proving it will be fact that possessing 5 integers (or rests) we are always able to choose such 3 of them with divisible by 3 sum. We'll use it twice in our proof. Here is the first time: we have 8 integers and taking any 5 of them we are able to select such a 3 of 5 with divisible by 3 sum. So 3 number are already forming a divisible by 3 sum other 8 – 3 = 5 number will be invited for selection of such 3 of them with divisible by 3 sum – this the second use of the argument we promised to use twice. So from these 6 integers we've formed two triplets with the divisible by 3 sums of numbers of each. So the sum of numbers of both these triplets is divisible by 3 too so completing the proof.

OTHER NATURAL BUT USEFUL GENERALIZATIONS

In the same book of Minsk contest (see [6] again) in senior forms a problem with finding 4 and 5 summands with divisible by 3 sums were proposed. It is so difficult to resist the temptation to formulate the problem even in quite general form:

At least how many integers L is it enough to take from the set of all integers from 1 till M so that it would be possible to choose such N integers among these selected L that the sum of chosen integers would be always divisible:

(A) by 10;
(B) by 100;
(C) by 2007;
(D) by 1 000 000;
(E) by a given integer K

We are keeping silence about more general problem.

Positive integers M, N and K are given. Determine the integer L = L(M, N, K) (depending on M, N and K) be the smallest number with the following property: selecting any L integers from the set all integers from 1 till M we are always able to choose N integers from these L selected numbers with the sum of all them dividing K.

All that we regarded was only some special cases of that. So we found out that L(100, 3, 3) = 5 and L(100, 6, 3) = 8

We asked also (using the notations we just accepted) just as it was in [6] about L(100, 4, 3) and about L(100, 5, 3). In our notations we asked also about L(M, N, 10), L(M, N, 100), L(M, N, 2007) and even about L(M, N, 1 000 000).

There not a few cases which could be characterized using words: **brave questions not so difficult to formulate, not very easy to solve.** Is it so also in our case?

CHAPTER XIII.
YET ONCE AGAIN ABOUT 100 CARDS

Once in selection contest (we remind that we already repeated in Chapter I well-known truth: **not the Saints (only) are forming pots**) the following problem was proposed:

We've found again 100 cards with all numbers from 1 till 100 one number written on one card. Two players in turn (the player who starts is told to be the first player) are taking these cards away from the

table – one card at a time till only 2 cards remain on table. Then the sum of numbers of these 2 remaining cards is estimated. If this sum is divisible by 3, then the 1st player is announced to be the winner if not - then the second one.

Again the question: Has any of them the winning strategy or is able to win independently what the other is doing?

From the psychological point of view this problem seems to be from these with an easy solution. We know of course that after we begun solving we can easily change our opinion to an opposite one.

To what conclusions could we came after 5 minutes of solving?

1. Situation described just above seems to be not so promising for the first player because his hopes for win are connected with the divisibility by 3 of sum of 2 remaining numbers and, on the contrary, the hopes for win for the second player is just the indivisibility by 3 of the same sum of these 2 cards they both left on the table.

2. The rests and not a numbers itself play at this place the most visible role. In order to make things as clear as a day let's reduce the number of cards from100 to10. We are sure that the reader didn't forget that this was called "reducing of impression of big numbers".

After these all these remarks we have to deal with situation with thrice as simple as it was. 10 cards are lying on the table: 3 with' 0's, 4 with 1's and 3 with 2's written on them. Two players in turn now will take away these cards one at a time till 2 cards remains on table. If the sum of these remaining two cards is divisible by 3 then the first is the winner if not then the second.

Is anybody from them able always wins independently what the other is doing? If anybody is able to win then it seems that rather the second one. We feel and are convinced that his chances are bigger. Indeed the chances of divisibility/indivisibility by 3 relates as 1:2 if the numbers are chosen at random. At any rate the chances of second player are bigger,

Let us make another usual for us thing and try to make one step back in order to see what a situation was before their last turn. The second player had already taken 3 cards and so was able, for instance to remove all three 0 cards corresponding to former cards 3, 6 and 9. So the second player can always achieve such situation that there are no more 0 cards lying on table but only cards with 1's and 2's.

Reasoning in such a way we understand that before the last move of both players the following 5 situation are with 4 cards left are possible: (A): all four 1's; (B): three 1's with one 2; (C): two 1's and 2's; (D) one 1 with three 2's; (E) all four 2's.

1 1 1 1
1 1 1 2
1 1 2 2
1 2 2 2
2 2 2 2

The situation now is easy to describe. If on the table

*there are only 1's or only and 2'*s,

then it is of no importance what will be taken away in last fourth move – in every case either two 1's or two 2's remain, their sum is indivisible by 3 and the second player clearly wins.

If on the table there are

three 1's and one 2 or, vice versa, three 2's and one 1, then the second must take another number as the first did and again either 2 1's or two 2's will remain and just as in previous case the second player wins.

Finally if there are

two 1's and 2's both

lying on table then in his last move the second must take the same number as the first did and again two cards with equal numbers remains.

Then we've got an answer, that it is the second player who has a winning strategy.

In the case with 100 cards as it was proposed in math camp the solution is identical – we are regarding again the situation before the last move of these players and everything what's told in the case with 10 cards could be repeated word by word.

By the way, in the camp not all participants solved this problem so it could be repeated:

It is not (only) the Saints who are forming (making) pots.

CHAPTER XIV.
MONOTONIC INTEGERS

Let's start straight from the definition of *monotonic integer.*

Definition. A positive integer **N** is said to be *monotonic integer* if there is a positive integer **M** such that the product of these 2 integers **N** × **M** may be written using only one digit.

For example a respectable number 12 345 679 is a *monotonical integer* because

$$12\,345\,679 \times 9 = 111\,111\,111.$$

Let's note that the last expression is often used for verifying whether the calculator is functioning reliably.

First natural question connected with such integers.

Could it be that every positive integer is *a monotonic integer?*

Indeed, all 1-digital integers

1, 2, 3, 4, 5, 6, 7, 8, 9

are clearly *monotonic integers* , If you have some doubts take **M = 1.**

Unfortunately this is the end of idyllically state because the next integer 10, which ends by 0 and no multiplication by any positive integer, can change it. From the other side the number **N** × **M** being product of two positive integers will contain non-zero digits too.

Learning from what we'd seen we ask: do there exist such an integer ending not by 0 which not a *monotonic integer*?

From the first sight it's not so clear is this question difficult or not.

Number 11 is at once monotonic and what about 12? How knowing nothing we could came to conclusion that multiplying it by 37 we'd get 444?

We'll try a following way. That one of numbers written below or

A, AA, AAA, AAAA, ...,

which divides 12 must divide 3 an 4. Then 444 would be a proper number because it's in clearest way divisible by 4. In the same time 444 is divisible by 3 because the sum of its digits 12 is divisible by 3.

And what is with 13 known under the name of devil’s dozen too? One of the possible attempts could be to take the number with no other digits but 1’s and proceed with the hope a long division. What’s will then happen? Just take a look.

$$
\begin{array}{l|l}
111111 & 13 \\ \cline{2-2}
\underline{104} & 8547 \\
71 & \\
\underline{65} & \\
61 & \\
\underline{52} & \\
91 & \\
\underline{91} & \\
0 &
\end{array}
$$

It’s no wonder that for this divisibility it is enough to take a number with 6 1’s, because we’ve do not forgot that

$$111\ 111 = 111\times1001 \text{ and } 1001 = 7\times11\times13.$$

And e.g. 155 – is it a *monotonic number*? You will probably wonder about the sources of our knowledge but we can ensure you that

$$3\ 584\ 229\ 390\ 681\times31 = 111\ 111\ 111\ 111\ 111$$

giving that

$$3\ 584\ 229\ 390\ 681\times155 = 555\ 555\ 555\ 555\ 555.$$

Now we ask what’s least 3-digital *monotonic number*? An answer is even simpler tan we could imagine if we are able to guess that …

We already know that 100 ending by 0 isn’t a *monotonic number* because as we’ve seen 0 in end of an integer is “ineradicable” when multiplying by any positive integer and in the same time 0 isn’t the only digit of the product because this product is a positive integer.

Next possible nominee is 101. Its decimal expression contains 0 but this isn't the last digit so we still hope that 101 is *monotonic*. This is indeed the case because

$$101 \times 11 = 1\ 111$$

indicating that 101 is the last 3-digital *monotonic number*.

We end this chapter **addressing** the reader with the following question.

The last digit of a number isn't 0. Can this number be not *monotonic*?

CHAPTER XV.
WHAT TO DO WHEN YOU DO NOT KNOW WHAT?

Now do something, won't you, my boy?

We would like to stress and remind you yours possible thoughts and emotions what bothers you when are in a similar situation as described in headline of that chapter. This is a kind of eternal problem and that state of mind is familiar to everyone independently how young or old a person might be or feel.

So what to do when you do not know what to do or how to start when you are not sure about your possible first steps?

This is a situation you are forced to deal with on each of your born days. Or at least on every second day or twice a week we suddenly find ourselves in situation in which we've never been before or we meet people, whom we've never seen before or we hear words that we never heard or are expected to fulfill the task we never did. Perhaps a similar feelings have a space travellers when being in the state of weightlessness – everything what's happens with them is so unusual and different from all they knew. This is curious, slightly scaring and so challenging that our mind and our entire being mobilizes for mastering of these situations all its powers and capacities, all energy, experience and common sense.

An accessible but nice mathematical problem allows us to model such kind of situations properly selecting a degree of complexity. They also allow us – and this is extremely important, especially when we are not lucky enough with the solving – to lay things aside (for some time) in order to return back later. As it was already mentioned the stage of these performances are located in one's mind and a whole human being is involved with all powers and will, with all sentiments and hopes.

Hereby we propose one such a problem which appears for us to be rather nice und unexpected and which is in the same non-standard and accessible for everyone who's not afraid to be bothered with it and knows what's scratched paper is.

Don't be afraid and too much impressed alone by the fact that this problem was once proposed on final round of Lithuanian mind contest as well.

We claim once again that this problem is accessible to everyone with is really familiar with a multiplication table. So we come over to

A PROBLEM ABOUT ONE SELF-CODING 10-DIGITAL NUMBER

and a reader will be immediately informed what does it mean to be self –coding.

We are to find a 10-digital integer which 1st integers indicates the number of 0's in its decimal expression, the 2nd – number of 1's, the 3rd – number of 2's and so on, finally, it last 10th digit indicates a number of 9's.

Literally we are asked to present one such number but really seeking for one such number we'll hope to understand how to find all such numbers – noblesse oblige!

At first the modest but essential or a question: in what way one such a number could be found?

If we found ourselves in an unknown city and are supposed to find there a person who speaks Abyssinian then we always have two possibilities:

1. To get all possible useful information about an expansion of the Abyssinian or even all languages all over the world;

2. To learn some Abyssinian words and start marching along the streets of that city repeating these words to everybody and asking whether he understand them.

We tend to an opinion that in mathematics as well as in an everyday life the second way is more fruitful. Will it be so in our case too?

Take any number for instance

$$1\ 111\ 111\ 111.$$

In the language on our problem if that number could prove itself to be a solution, then its first digit indicates that there is one 0 in its decimal expression and this stops us at once because our number 1 111 111 111 has no zeros. With this concrete observation we understood somehow more deeply that if a number is suitable and 10-ditigal then:

that number necessary contains some zeros.

Let's try another more irregular number already with zeros, say,

1 234 564 089.

This integer is even "more unsuitable" because in the case that it were an answer we would have that this number contains 1 zero, two 1's, three 2's, four 3's and five 4's in its decimal expression. But we can't have so many or already at least 1 + 2 + 3 + 4 + 5 = 15 digits in a 10-digital integer.

Now we understand that if a 10-digital number is an answer of our problem then

Digits of that integer are rather small because sum of them all must be 10 so its decimal expression is "rich on zeros".

After noticing that potential richness on zeros it's appears quite natural to initiate sorting by possible number of zeros in the decimal expression of the number in question starting from above.

What's the highest number of zeros in that expression? Again because it's a 10-digital number so its first digit isn't zero and **it can contain at most 9 zeros**. But then an expression must start with 9 because zeros is coded by first digit which must be 9 and all other integers must be zeros as it guarantees that 1^{st} digit 9. But then it is an integer

9 000 000 000.

But this leads to contradiction because the 9 must be coded by last integer and consequently this last integer can’t be 0 and must at least be 1 and we no more have 9 0’s in the decimal expression of the number.

Then consequently we consider **case with 8 possible zeros**. If there are 8 zeros that mean that the 1st digit is 8 and last but one digit is 1. Because the sum of digits is 10 so there somewhere is another digit 1. This means already two 1’s in that expression. But then this number as such having two 1’s looks like

8…1…10.

Contradiction now is induced by the fact that the self-coding number with two 1’s must have 2 as its second eldest digit which again contradicts to the fact that the sum of all digits of that number is 10.

Further on the case with 7 zeros or with the first digit being 7. This 7 must be coded with 1 as the 3rd digit from the end – it can’t be two 7’ in that number! So it’s an integer of the form

7 … … 100

with some integers in the non-indicated positions.

Then if that indicated 1 is the only 1 at all it must coded with another 1 (in second eldest position) and the number then looks like

71… … 100

which is already a contradiction.

If that number contains at least two 1’s and consequently its second eldest digit is at least 2. This is again the contradiction because the sum of digits is then more than 10 because it is at least

$$7 + 2 + 1 + 1 = 11 > 10.$$

The case with 6 zeros follows. Then the 4th digit from the end is 1 and again this 1 can’t be only 1 in that expression. So similarly there are again at least two 1’s in its decimal expression and again these two or more 1's must be coded by at least 2 in the second eldest digit leading to the possibility

6 210 001 000

which proves itself to be the only proper example of such a self-coding number.

It ought to be strictly proved that this is the only possible answer of such self-coding integer. Of course it is necessary getting an answer in some step to write equalities. But when we are starting not from regarding of some concrete examples but from writing equalities we must remember that that way is more abstract and consequently more difficult one.

A concrete experience is almost of an indispensable value.

This could serve as partial explanation why at school the situation with the solving of combinatorial tasks is not so simple and cloudless and the achievements on that field are not as high as they could be. The possible concrete approach and brave regarding of simple(st) special cases may also explain why the students in lowest grades have often better results as these they have finishing already their high school career. The former hearing the problem often already immediately wishes to know which kind of formula is there to be applied instead of regarding at first some special cases so gathering an experience and making natural but so important observations. After this highly useful preparation the application of suitable formula if necessary is usually done without any mistakes.

CHAPTER XVI.
ANOTHER 10-DIGITAL ADVENTURE

It is known that a strict chief is not very easy and sometimes almost impossible to please. His people often go along repeating that a chief would is not bad only his demands are too high.

Let’s take a look at the problem which demands from a 10-digital number so much that it seems that no 10-digital number can fulfill all these conditions.

Does there exist such 10-ditigal number *ABCDEFGHIJ* all 10 digits of which are different and such that *A* is divisible by 1, number *AB* formed by first two its digits is divisible by 2, number *ABC* formed by its first three digits is divisible by 3 and so on, finally, the number *ABCDEFGHI* is divisible by 9 and this 10-digital number *ABCDEFGHIJ* itself is divisible by 10.

First observations:

1. If the whole number is divisible by 10 then it ends with zero ($J = 0$).

2. If *ABCDE* is divisible by 5 then its ends by 0 or by 5. Because 0 is already used as the last digit then $E = 5$.

3. There is no need to take care of *A* which can be any integer.

4. Because last digit $J = 0$, then the sum of digits of number *ABCDEFGHI* is equal to a sum of all digits from 1 till 9 or is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$ which is divisible by 9 that means that also this 9-digital number is always divisible by 9.

In other words there no need to make troubles about the divisibility by 9 as well.

5. Second, fourth, sixth and eight digits *B*, *D*, *F*, *H* of the number ***ABCDEFGHIJ*** must be even because of the divisibility of corresponding fragments by 2, 4, 6 and 8. That's why digits on all remaining so-called odd places or its first, third, fifth, seventh and ninth digits must all be odd. Each even digit as the second digit automatically guarantees the divisibility of fragment *AB* by 2. Because its third digit is odd then the fragment *ABCD* is divisible by 4 only if *D* is either 2 or 6.

Let us recall the criterion of divisibility by 8:

An integer is divisible by 8 if a number formed by its last 3 digits is divisible by 8.

For example 2016 is divisible by 8 because 16 is divisible by 8.

Because the fragment

ABCDEFGH

must be divisible by 8 and its sixth digit is even then for the divisibility by 8 it is enough that the fragment

GH

would be divisible by 8. Keeping in mind that the 7th digit must be odd we conclude that *GH* might be possibly 16, 32, 72 or 96 (we must exclude 56 from our consideration because 5 is already "engaged" as a 5th digit)

Note that the fragment *ABC* must be divisible by 3 and *ABCD*5*F*- by 6, so consequently fragment *D*5*F* is divisible by 3 as well and because *ABCD*5*FGHI* is divisible by 9 hence *GHI* is also divisible by 3.

If *GH* = 16 then the only possibility for *DEF* is to be 258 and our number looks like

*ABC*25816*I*0

leaving the possibilities for *I* be 3, 7 or 9 but neither 163 and 165 nor 169 are divisible by 3. It *GH* is 96 then after similar reasoning we would conclude that there such two numbers

1472589630 or 7412589630

left for further considerations. Unfortunately their 7-digital fragments

1472589 and 7412589

are not divisible by 7.

Further on *GH* being 32 wouldn't lead us to the wanted 10-digital number so it remains for our consideration only the case *GH* = 72 leading us to the only answer

3 816 547 290.

We could notice that in this 10-digital integer digits in even places are decreasing and in odd places - on the contrary - increasing if only 1 and 3 could be changed.

CHAPTER XVII.
ORDER LIKE THAT IN A DICTIONARY

"Let us take them in order…"

After Lithuania entered EU all possible matters connected with its cultural, spiritual and intellectual heritage additionally gained on importance.

Each country entering into closer relationship with other countries is eagerly interested to remain attractive for others seeking to understand more about the essence of its own as well and also obliged to try to spread more that spirit to the neighbours.

So we have a proud and important word

L I E T U V A

which in Lithuanian means of course Lithuania. By the way it ought to be added that the Lithuanian language is very archaic Indo-European language and is told to be similar to Sanskrit. Lithuanian language together with Latvian and Prussian languages belongs to the subgroup of Baltic languages.

In the original word for *LITHUANIA* or in the word *LIETUVA* there are 7 different letters. After Lithuania entered an EU the special vocabulary with all possible permutations of these letters in a lexicographical order - as it's used in every dictionary - was edited in Brussels by an by the Lithuanian section of the Association of Baltic friends.

In connection with that the following questions were to be cleared:

1. In order to keep a principle of problems structuring it was proposed to start with the most natural and perhaps easiest question: lexicographically which word in that dictionary would be the first?
2. Which words would be the 2^{nd} and the 3^{rd} ones?
3. What is the number of all possible permutations (or words according to the philological terminology)?
4. Which word would be the 2007^{th} one?
5. In what place would we find the word *LIETUVA* itself, the word that gave an impulse to the Association to prepare that edition representing all possible permutations of the word *LIETUVA*?

A N S W E R S

1. Ordering the words of *L I E T U V A* lexicographically we have that the A is the 1^{st} letter, E – 2^{nd}, I – 3^{rd}, L- 4^{th}, T – 5^{th}, U – 6^{th} and finally V - the last or 6th letter. **Hence in vocabulary containing all permutations of the word *LIETUVA* the very first word would be of course**

A E I L T U V.

2. The pleasure to indicate the 2nd and 3rd words in this dictionary if all permutations **is left to the reader.**

To all these who are willing to ensure themselves that they understand these matters properly for the sake of completeness we would like to remind or repeat some simple(st) things..

Having only two letters A and E we would get 2 possible permutations

AE and *EA*.

Similarly we would get 2 permutations *EU* and *UE* from the abbreviation *EU.*

Analogically possessing 3 letters *E*, *U* and *I* than ordering permutations lexicographically we would get already 6 possible permutations

AEI, *AIE*, *EAI*, *EIA*, *IAE*, *IEA;*

exactly as it would be 6 permutations *ASU, AUS, SAU, SUA*, *UAS*, *USA* what may be got from the letters of an official abbreviation *USA* of United States of America.

Similarly having 4 letters (*A*, *E, I*, *L,* if would choose to use our letters) our 4th letter *L* could be written in front of 6 possible permutations of letters *A*, *E* and *I* giving the first 6 permutations of 4 letters *A*, *E*, *I* and *L*. Another 6 permutation of these 4 letters we'll get writing the fourth letter *L* between the first and second letter of all 6 permutations of letters *A*, *E, I*. Further 6 permutations we would get writing *L* between the second and the third letters and the final sixth - writing *L* after these 3 letters in 6 possible permutations of them. All this gives us

$6 + 6 + 6 + 6 = 24$ possible permutations of already 4 different letters.

So if with 2 letters there are

$$2 = 1 \times 2$$

permutations or 2 different possibilities to order any 2 objects, then with 3 letters there are already

$$6 = 1 \times 2 \times 3$$

permutations or 6 different possibilities to order any given 3 objects, then taking 4 letters we have

$$24 = 1 \times 2 \times 3 \times 4$$

possible permutations or 24 possibilities to order any 4 objects. After we’ve stated this the suspicion arouses that taking any 5 subjects we’ll have

$$1 \text{ x } 2 \text{ x } 3 \text{ x } 4 \text{ x } 5 = 120$$

possibilities to order them, further on taking 6 subjects we would get

$$1 \text{ x } 2 \text{ x } 3 \text{ x } 4 \text{ x } 5 \text{ x } 6 = 720$$

possibilities of ordering these 6 subjects.

These suspicions turn out to be right. More general, there is

$$1 \times 2 \times 3 \times \ldots. \times (n\text{-}1) \times n = n! \text{ (called } n \text{ factorial)}$$

possibilities of ordering n given subjects and **especially there are**

$$\mathbf{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 = 720 \times 7 = 5040}$$

ways to order 7 given subjects or also all 7 letters of the word *L I E T U V A*.

This is also an answer to part 3 of our question.

When n is increasing n! is growing up very rapidly. So if we would like to count in how many ways could 10 students stay in the queue waiting for tickets to the performance of “The Beatles” we would get that there are

$$10! = 1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10 = 5\,040\cdot8\cdot9\cdot10 = 3\,628\,800$$

possibilities of different orderings when waiting for tickets (this number is approximately equal to the number of inhabitants of Lithuania).

Concerning the answers to 4 and 5. Indicating the 2007th word of our dictionary and looking to the place of word *LIETUVA* in that dictionary let's say some words about the structure of this edition. This dictionary is divided into 7 chapters named after their first letter correspondingly Chapter *A*, Chapter *E*, Chapter *I*, Chapter *L*, Chapter *T*, Chapter *U* and Chapter *V*.

Chapter *A* include the words from 1st till 720th, Chapter *E* – from word 721 till 1440, Chapter *I* contain words 1441 – 2160, Chapter *L* - 2161 – 2880 words, Chapter *T* – 2881 – 3600, Chapter *U* – 3601 – 4320 and Chapter *V* contains words from 4321 till 5040.

So we with our 2007th word land in Chapter *I* and with word *LIETUVA* – in Chapter *L* of course.

Now the each Chapter is divided into corresponded subchapters. So the 3rd Chapter *I* is divided into 6 subchapters *IA*, *IE*, *IL*, *IT*, *IU* and *IV* (and correspondingly 4th Chapter *L* is divided into 6 subchapters *LA*, *LE*, *LI*, *LT*, *LU* and *LV*). Each subchapter counts 120 words.

It is very easy to find out that that the 2007th word will be found in subchapter *IU*, which contains word from 1921 till 2040 (and correspondingly *LIETUVA* is to be found in the subchapter *LI* containing words from 2401 till 2520.

Each subchapter will be divided into 5 sections with 24 words in each subsection. In the subchapter *IU* there will be 5 sections *IUA*, *IUE*, *IUL*, *IUT* and *IUV* (and correspondingly in the subchapter *LI* there will be 5 sections *LIA*, *LIE*, *LIT*, *LIU* and *LIV*). 2007th word will be in the section *IUT* containing words from 1993 till 2016 (and correspondingly *LIETUVA* in the section *LIE* containing words from 2425 till 2448).

Further on each section will be divided into 4 subsections with 6 words in each.

So section *IUT* will include 4 subsections *IUTA*, *IUTE*, *IUTL* and *IUTV* (and correspondingly section *LIE* will include 4 subsections *LIEA*, *LIET*, *LIEU* and *LIEV*). 2007th word is in subsection *IUTL* with other words from 2005 till 2010 (correspondingly *LIETUVA* is in subsection *LIET* with other words from 2431 till 2436).

Now subsection *IUTL* contain the words from 2005 till 2010 which are:

IUTLAEV, *IUTLAVE*, *IUTLEAV*, *IUTLEVA*, *IUTLVAE* and *IUTLVEA*.

So the extended answer to the part 4 sounds as follows: 2005th word in that edition of Baltic friends is *IUTLAEV*, 2006th is *IUTLAVE* and finally 2007th word or direct answer to the part 4 is *IUTLEAV*.

Similarly subsection *LIET* contain the word from 2431 till 2436 which are:

LIETAUV, *LIETAVU*, *LIETUAV*, *LIETUVA*, *LIETVAU* and *LIETVUA*.

So the answer to part 5 is that *LIETUVA* will the word number 2434.

PROPOSALS.

Form and solve analogical problems for Society of ICELANDIC and for the Society of LATVIAN friends.

1. Which EU countries would have the least number of pages and how many items would be there are in such dictionary?

2. With EU country would possess the largest dictionary of the kind and how many items should be included in it?

CHAPTER XVIII.
NUTS AND DIVIDING OF CHOCOLATE

Imagine that we are asked by our best neighbour John Cleverest to advice him to deal with following problem which he found in some Scandinavian book. Its formulation is the following.

We possess a chocolate bar which is in the form of square and of size $n \times n$ which is lined up in usual way into n^2 unit parts of size 1×1. On some of these 1×1 parts LILLEBROR may put a nut. Afterwards CARLSSON breaks this chocolate bar along these indicated lines into two rectangular parts.

The question which Mr. Cleverest is presenting seems to be rather serious and interesting: at least how many nuts must Lillebror locate on that plate of chocolate that after this plates is broken in the usual way along these indicated lines into two rectangular parts at least on one part there are at least n nuts?

From psychological point of view we may be impressed here by “great number complex”, at the same time the situation such as described in this problem is not standard so that we need some time to get used with that situation in order to be able to draw some right conclusions from it.

Let us drastically simplify the situation taking a chocolate bar of size 2×2 with the hope that even in such a simple partial case some useful observations could be made.

So we a dealing with question: at least how many nuts ought to be lied on some 1×1 parts of 2×2 chocolate bar so that when chocolate is divided in two rectangular parts at least on one part there are at least 2 nuts.

This case 2×2 is easy to consider and it is clear that to put 3 nuts it would be enough – nuts will be naturally denoted by N.

N	
N	N

Wouldn't be enough to put on a bar 2 nuts only?

The answer is: no, it wouldn't. But it would be suitable to motivate our answer.

Assume that 2 nuts were enough. Then if both these nuts are located on the same row or column, then we can break a plate between them so separating these nuts. If they not in the same row and in the same column then dividing the plate in any way we'll separate these nuts anyway.

Psychologically it's not so easy for us to convince ourselves that even is such a simple situation we need some kind of proof that 2 nuts it's not enough. But to think correctly regarding all possible cases is absolutely necessary – otherwise we could miss the truth which from human and logical point of view is the worst thing that may happen.

In other words if we seek for treasure knowing for sure that it is in a flat with 100 rooms and we didn't found it in 99 rooms that we must never forget that this treasure may be hidden in that 100th smallest room or possibly even in the kitchen.

We are going to regard bigger bars of chocolate – meanwhile dealing with that one of size 3×3 and repeating our question: **at least how many nuts must we put on that 3×3 plate now that after division of it in two rectangular parts at least one part would be with at least 3 nuts on it?**

We gained already a bit on experience in order to understand 5 nuts for 3×3 plate is always enough for we can locate them e.g., "in cross" as is shown below:

	N	
N	N	N
	N	

By the way it ought to noticed that 5 nuts for 3×3 bar will always work with each location of nuts because we are putting at most 1 nut on each 1×1 part of our bar – so dividing the bar with 5 nuts on it in to two rectangular parts we always will have at least 3 on one of these parts. The same would be right with 3 nuts on 2×2 chocolate bars.

Now back to the question: well, 5 nuts will always do; what about 4 nuts? The answer is affirmative: in the above shown picture we can take away any nut we may choose and remaining 4 nuts will do. These two principle cases are illustrated in the pictures:

	N	
N		N
	N	

	N	
	N	N
	N	

And what would happen if we'd try to take away yet one more of nuts. Will 3 nuts do? No, 3 nuts wouldn't do. Why?

We will again motivate it. It will completely similar as it was in the case of 2×2 chocolate bars:

1. If we find any 2 nuts in one row (one of such cases is shown in picture below) then we divide the bar in two rectangular parts broking the bar “between” them and so separating the nuts.

N		N

2. If we find 2 nuts in one column then again we divide the bar in two rectangular parts broking “between” the nuts again separating them.

	N	
	N	

3. If there no such row and no such column with at least a pair of nut in it then all these 3 nuts are located in different rows as well as in the different columns (one of such possibilities is been demonstrated below) then any possible partition of bar in two rectangular parts will separate one of nuts from remaining 2.

N		
		N
	N	

Now it would be probably the high time to eat a real chocolate bar with rich supplies of nuts on it naturally in the company of friends with no fear that after division these friends occasionally will take not the smallest piece of the bar. That proves that our bar is of highest quality and that the preferences of our friends are corresponding – they understand what they choose. Are there no more pieces of chocolate left? That's good so. Let's try going on "testing and tasting" nothing more suitable but simple and accessible problems.

CHAPTER XIX.
ADVENTURES WHEN DEALING WITH BIGGER BARS

Trying to remain consistent and consecutive we are supposed to say some words concerning 4×4 chocolates bars. In that case everything runs without any complications.

Firstly we simply state that if we will arrange a "cross" of 5 nuts on the 4×4 bar as shown below then it convinces us that 5 nuts is enough:

	N		
N	N	N	
	N		

That with fewer nuts we'll achieve nothing can be proved by repeating word by word that what was told in the cases $n = 2$ and $n = 3$.

What can be expected on the 5×5 bar?

Keeping in mind all previous cases we understand well enough that we'll need at least 6 nuts – again that 5 nuts ate too few is more than clear. We will try to put 6 nuts on the 5×5 chocolate bar in such a way that dividing bar in any way in two rectangular parts on one of parts at lest 5 nuts could be found.

But in that case somehow we are in no way successful. This is bad; moreover it is more than bad. We do not want to lose such a nice law, such a simple and that's why so nice rule expressed by words:

On n×n bars exactly n+1 nut will always do.

That on 5×5 bar 7 nuts is enough shows their dislocation in the picture below:

	N	N		
N	N	N		
N	N			

Again all attempts with 6 nuts are leading nowhere. Perhaps that is only us who are not able to realize it? Perhaps others would be able to do that?

It would be really a pity to lose such a nice rule. What to do? How to behave? Where to apply? In that case of 6×6 bars our reliable friend Mr. Computer would of course solve all problems simply and quickly running over through all possible cases. That would so in cases n = 6, n = 7 and so on but usually Mr. Computer can't help running over all possible cases for n being arbitrary large.

So we are almost forced to begin with the abstract theoretical considerations that every attempt to locate 6 nuts on 5×5 bar would fail. So once again this ever sounding and everlasting eternal question: "Why?" or again to

Reductio ad absurdum as a source of our logical force

"The method employed I would gladly explain,
While I have it so clear in my head."

A proof that's following our thoughts and considerations or with other words: "Attention,

please! You’re trying to prove something – be extremely careful!”

We would like to ask everyone to help us to ensure the clearness of proof and we apply to our readers at first to be our guards strictly following that way.

Assume that having only 6 nuts nevertheless it is possible to locate them on the 5×5 chocolate bar putting at most one nut on some 1×1 lined parts of chocolate bar in so clever way that dividing the chocolate in any possible way in two rectangular parts we will always find at least 5 nuts on one of divided rectangular.

Let’s now do the following. Firstly prepare 4 such a chocolate bars with that cleverest location of nuts on it. Now we will do the following 4 divisions of these identical bars.

1 division will be the following one: we divide the bar in two parts or rectangular – the **left** one and the **right** one - in such a way that the **left rectangular is the smallest possible rectangular with already 5 nuts on it.** We will refer to that smallest left rectangular as to part A will and lay it aside.

2 division will also be a division in two rectangular: **right** one and **left** one in such a way that **right rectangular is smallest rectangular with already 5 nuts on it.** This smallest right rectangular will get the name of rectangular B and we’ll lay it also aside.

3 division will be a division in two rectangular parts: **top** part and **bottom** part in such a way that **top rectangular is smallest possible rectangular with already 5 nuts on it.** This smallest top rectangular from now on will be named rectangular C – lay it also aside.

4 partition will again be a partition with top and bottom rectangular with the only difference that **now the bottom rectangular is smallest possible rectangular with already 5 nuts on it.** Let this smallest bottom rectangular be rectangular D and lay it also aside.

Now we intend to prove that the rectangular A and B have exactly one column in common, rectangular C and D – one row in common and all these 4 rectangular – the only common 1×1 field.

The proof that the (left) rectangular A and (right) rectangular B has exactly one column in common run like follows: assume that this not so, then there are two possibilities to b e considered:

- A and B has no common column;

•• A and B has at least two columns in common.

When (•) takes place then a rectangular A and B having no common columns doesn't intersect at all and so the union of them both being disjoint and so doesn't exceeding the initial chocolate has already at least $5 + 5 = 10$ nuts which contradicts the assumption that for the whole initial bar 6 nuts were enough.

When (••) takes place then we take the first column from the right side of their intersection and remove it from rectangular A. Then that lesser rectangular LA doesn't contain 5 nuts because the bigger rectangular A was the first from left rectangular containing 5 nuts.

Now regard the partition of the chocolate bar consisting from the rectangular LA and its complement CLA, which is one from right rectangular but not the biggest from right rectangular hence also not containing 5 nuts.

But that clearly contradicts the fact that any partition into two rectangular parts gives at least 5 nuts on one of these parts.

Exactly in the same way repeating word by about rows what it was told about the columns we would get that rectangular C and D has exactly one row in common.

Then consequently all these rectangular A, B, C and D have exactly one 1×1 common field.

Now we'll apply once again the double counting.

All these rectangular A, B, C and D together have at least 5 + 5 + 5 + 5 = 20 nuts.

Now each nut from that bar may belong to at most 3 of 4 rectangular A, B, C, D, except of possibly one nut, which may lay on that one field that is common to all these 4 rectangular.

So at most 5 of these 6 nuts may be counted thrice and the sixth nut – possibly 4 times indicating that on these 4 rectangular may be at most 3 × 5 + 4 = 19 nuts.

But we words "at least 20" and "at most 19" when related to the same subject contradict each other.

So 6 nuts is not enough for 5×5 bar.

We would like to draw your attention that similar considerations are also fruitful with bigger chocolate bars.

We also highly recommend to regard some intermediate cases, e.g. n = 10.

The idea itself and the special case with n = 100 was proposed at Sankt-Petersburg Contest (see [8], problem 52, p. 54).

CHAPTER XX.
WHAT TO DO WHEN IT'S NOT SO WELL UNDERSTANDABLE HOW TO LESSEN A HUGE NUMBER?

One (classical) case: it may happen that there are no possibilities for this.

The reader may reply citing all advices to lessen all these huge numbers without losing the essence and intrigue of proposed problem. But what ought we to undertake in cases when it's impossible. As an example we could regard another problem of Sankt-Petersburg contest A.D. 2002 (see [4], Problem 39).

Show that from any 10-digital numbers with no zeros in its decimal expression it is always possible to "cut out" such a fragment formed by 3, 4 or 7 its consecutive digits, which is divisible by 3.

We do not know in what way these numbers could be reduced? 10-digital numbers are billions or thousands of millions - there are probably too much of them to be regarded one by one even by the computer.

But understand how it all could be arranged is challenging, exciting and interesting.

By the way, awakening of curiosity in problems solving is extremely important matter of psychological nature. Nice formulations may attract the potential solver to start dealing with a problem. On the other hand clumsy formulation of problem may often push away a potential solver or not attract him which is almost as bad.

So the involving of potential solver in the process of seeking the truth is the problem of the first-rate importance. This long-fallow land still remains relatively uncultivated till our days.

SIMPLE AND NOT BAD ADVICE WHICH CAN PRACTICALLY ALWAYS BEEN APPLIED

This advice is: regard concrete examples and see what you can learn from them. We understand that one suitable example can't give us all essential information about the essence of problem but an example is always an indispensable source of useful information about the nature of our investigations.

Taking the first possible 10-digital number such as

4 357 892 183

We state that its first 3-digital fragment 435 is already divisible by 3. What else have we seen in this example more precisely? We have seen that each such 10-ditigal number has 8 three-digital, 7 four-digital and also 4 seven-digital fragments. The condition "with no zeros in its decimal expression" guarantees that all these fragments formed by corresponded 3, 4 or 7 digits are indeed 3-, 4- or 7-digital integers.

Afterwards we can apply some almost unnoticeable but rather convenient simplification replacing all digits of 10-digital integer in such a way: 1, 4 and 7 will be 1; digits 2, 5, 8 will be 2; digits 3, 6 and 9 will be 3 – usual replacement modulo 3. So from now on we are to deal with 10-ditigal integers, with decimal expression is formed exceptionally by digits 1, 2 and 3.

What could follow afterwards? Many things, for instance also another regarding of another "concrete" 10-digital number, say,

2 212 233 221.

This number doesn't suit immediately because all its 3- and 4-digital fragments turn out not to be divisible by 3. Really the sums of digits of all its 3-digital fragments are

5, 5, 5, 7, 8, 8, 7 and 5.

Sums of corresponding 4-digital fragments are

7, 7, 8, 10, 10, 10 and 8.

Now it's only the sum of digits of 7-digital fragments left for consideration but now already the 1^{st} such sum or

$$2 + 2 +1 + 2 + 2 + 3 + 3$$

is equal 15.

It remains to notice something essential else in order to be able to finish with the proof – we have a strong feeling of moving in right direction.

Let's start for the finish.

Take the first 7-digital fragment *ABCDEFG* of that 10-digital number

ABCDEFGHIJ.

If this fragment *ABCDEFG* is divisible by 3 there is nothing to do because everything is already done.

If this fragment isn't divisible by 3, then this rest is either 1 or 2. Let us in two different ways "split" this fragment in two fragments: firstly in

ABC and *DEFG*

and secondly

ABCD and *EFG*.

None of these 4 fragments is divisible by 3 because otherwise everything would be done again.

Let us now regard two essential cases of indivisibility of 7-digital fragments ABCDEFG by 3 with the rest being:

(•) 1 or **(••) 2.**

In the case (•) the rest of division of fragment *ABC* by 3 can be only 2 (otherwise this rest being 1 implies that the "remaining" fragment *DEFG* is divisible by 3) so the rest of remaining fragment *DEFG* is also 2. Exactly from the same reasons the rests of the division of division by 3 of remaining two fragments *ABCD* and *DEF* are also 2.

Now we have got that both fragments *ABC* and *ABCD* have the same rest 2 when divided by 3. This implies that $D = 3$.

In the case (••) when the rest of division of fragment *ABCDEFG* is 2 and splitting it in the same fragments *ABC*, *DEFG*, *ABCD* and *EFG* we would get that their rest when dividing by 3 now is always 1 and then repeating word by word our previous argumentation we would get again that $D = 3$.

Taking another 3 possible 7-digital fragments *BCDEFGH*, *CDEFGHI* and *DEFGHIJ* we would get again that their "middle" digits or correspondingly integers *E, F* and *G* are also 3 (three integers *D, E* and *F* would be already enough). So we have proved that *DEF* is 333 so clearly divisible by 3.

Let us repeat what was proved: *from any 10-digital number with no zero digits in its decimal expression it is possible to cut out 3- or 4- or 7-digital fragment which is divisible by 3.*

Giving no guarantees at all we propose to our reader to find the answer to the following question:

Not all digits of a 6-digital integer are equal. Can we always possible choose such 2 or 3 its consecutive digits such that the 2- or 3-digital integer formed by them would be divisible by 3?

What about the same question when all digits of a 6-digital integer are different?

CHAPTER XXI.
THE NOWADAYS CHALLENGES OR CONCERNING MARKET PSYCHOLOGY

Curiosity killed the cat – satisfaction brought it back.

If we really intend to awake the human curiosity we may propose some from the first glimpse quite realizable or life-similar situations accompanied by the eternal question: is it really so or that is only an illusion? We are always eager to regard and give the answer to the question: **what could be the extreme cases in the given situation** (extreme often means rather unusual) **and are they realizable?**

It seems that in some Belarusian creative problem books we have seen a problem which we would like in an adopted form to use in this chapter. We will try to present it in a form of reminiscences of an old and experienced business-man. There are different views concerning the use of such adoption from enthusiastic till extremely skeptical ones. All these views are highly understandable but the experience of the author shows if these adoptions are done in at least sati factionary way then they have proper place and prove their value. This adoption like many others since A.D.1998 had been published by the author in series of articles in the Lithuanian Computer Magazine.

– Business is a difficult field to start, - told the old business lion to his young colleagues starting the lecture “Towards the optimism in business” and continued.

– Remembering the first year on my activities I must frankly confess: **in that year my total expenses of every 5 consecutive months were higher as incomes or, once again, a balance of every 5 consecutive months was negative during all that first year.**

– Why are you speaking then about optimism? - came the replica from the audience.

– Because the balance of my whole business of the first year was positive – or my total incomes exceeded my total expenses, - was the reserved answer of the lecturer.

Is this possible?

At this place we are chanced to meet some psychological difficulties connected with the necessity to distinguish between the usual or typical cases which we meet every day and these who seldom happen and are known under name of very special or untypical cases.

In problems solving having that in mind we often give *an advice to the solver to regard the worst or most inconvenient cases*.

In the above mentioned business situation the most usual situation is of course the following one: if the expenses of every 5 consecutive months during whole year are higher than incomes then the total balance of year is usually also negative.

In most cases it's indeed so but not always and not in each case.

In the table shown below first row indicates the corresponding months and the second row – its balance.

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
2	2	2	2	-9	2	2	2	2	-9	2	2

It is obvious that according to that data the balance of every 5 consecutive months is negative (and always -1) and yet despite of that the balance of the whole year is 2 so positive as stated.

CHAPTER XXII.
MORE ON CHARM OF CONCRETE NUMBERS

In English books of creative problems we chanced to meet a problem which we'll present in slightly structured form or trying to engage the reader gradually or step by step to its solving.

At the beginning we are looking for any positive integer the decimal expression of which contains only 2's and 3's – both digits are present – und which is divisible by 2 and also by 3.

The text reminds us about the criteria of divisibility by 2 and by 3 – they are known really to each student of younger grades already: a number is divisible by 2 if its last decimal digit is even and by 3 of course iff (iff stands for the abbreviation "if and only if") the sum of its digits is divisible by 3 – we've applied it already lots of times.

So that the last digit of such a number we are looking for or the one which is formed only by 2's and 3's must be 2. For the divisibility by 3 the sum of digits must be divisible by 3 (let's recall again that both ciphers are present). If we take 3 copies of number 32 with the sum of digits 3 + 2 = 5 and put them together we'd get the "composite" number

323 232

which is indeed divisible by 3 because its sum of digits is 5×3 = 15 and is also divisible by 2, because it last digit as it was already mentioned is even.

Now after we've already found one such a number it would be natural to ask about the least such a number.

The last digit of an earlier found integer must be 2 and our task now is to collect a divisible by 3 sum of its digits. The wanted number containing at least one digit 2 and at least one digit 3 will have the sum of digits at least 5. The number next to 5 which is divisible by 3 is 6, but 6 isn't a sum of some 2's and 3's in the case when both of them must be present. Our next hope would be then 9 and this is not vain hopes because it is indeed possible to gather 9 adding three 2's with one 3. In order to get the least such number we must locate that only digit 3 possibly near to the end of the number – in our case as the digit of tenths getting the number

2232

which is the smallest possible integer among all such an integers.

Now we propose to the reader a similar problem only with two other digits, namely, 8 and 9: **determine at first one such integer whose decimal expression contain only 8's and 9's (again both sorts of integers must be present) and which is also divisible by 8 and 9 and further we naturally ask to determine the least such an integer.**

Such problem could be raised with any pair of ciphers just as it was with the mentioned pairs (2, 3) and (7, 8). The problems with concrete numbers are always interesting because it is understandable what we expected to do and in the same time it is more or less clear in what way all this could be achieved.

An additional interest to our actually chosen task adds a circumstance that there is no commonly approved criterion of divisibility by 7. With divisibility by 8 things looks much better: it is well enough known *that a number is divisible by 8 if only if the number formed by its last 3 digits is divisible by 8.*

Again its last digit must be 8 otherwise the number wouldn't be even. Further its second digit from the end must also be 8, otherwise the number would end by 78 and wouldn't be divisible by 4 and this is too bad because we need more than the divisibility by 4. In a similar way its third digit from the right must then also be 8 because otherwise the number would end by 788 and as such wouldn't divisible by 8. Indeed if it were so then also

$$12 = 800 – 788$$

would be divisible by 8 which is clearly not the case.

Our number must end with three 8's or 888 and we are expected also to use at least one 7. That mean another smallest candidate could be 7 888. But 7 888 is not divisible by 7, because 888 being equal to the product $2^3 \times 3 \times 37$ is not divisible by 7. The next candidate would be then a number 78888, but this number isn't again divisible by 7.

Following that way we would soon find the 7-digital number

7 888 888

which is smallest among all such numbers.

CHAPTER XXIII.
YET ANOTHER TWO NICE PROBLEMS WITH CONCRETE INTEGERS

More than 10 years ago in some Sankt-Petersburg's book of creative problem we've found such a nice problem with no x mentioned in the formulation.

Prove that an integer

$$40 \times 66 \times 96 + 53 \times 83 \times 109 \ (= 732\,931)$$

is not prime number (e.g. has some other divisor different from 1 and 732 931).

For computer this problem would be of no interest because the number in question is even smaller than a million and checking whether is it prime or not would last only small parts of second.

We confess that for us sometimes or frankly speaking rather often is very pleasant to set against the computer our human skill, experience and invention. As a simpler exercise of that kind with the same idea of solution we may propose another problem with smaller number involved.

Prove that

$$23 \times 27 \times 29 + 50 \times 52 \times 56 \ (= 163\,609)$$

isn't prime number.

Even more simple but still saving the same idea would be the problem whether the number

$$9 \times 13 \times 15 + 38 \times 40 \times 44$$

is a prime number?

We would like to encourage the reader solve these questions without asking for any help of computer. We would like only to mention that all these 3 examples are of the same "structure" or "similarly build" and that this could play some role. Other details including the pleasure of solving it are kindly left to the reader.

If you will employ nevertheless a computer for the finding of a proper divisor of number in question then after seeing what a smallest divisor we get we will have some clear ideas what could we notice before applying a computer.

We would like to pay once again your attention to the fact that all these numbers is a sum of two summands each of these summand being product of 3 integers and these two tripled of product numbers are somehow are somehow clearly related to each other.

YET ANOTHER NUMERICAL PRELUDE

In Hungarian wisdom books we have seen the following prelude.

Prove that the number

$$5^{12} + 2^{10}$$

isn't prime number e.g., has a divisors different from 1 and the number itself.

Again we must frankly state: this problem is not for the computer: such numbers for him seem to be amazingly small ones. And for us the following question is always rather challenging and exciting – how such a problems could be mastered without any calculator using only human invention multiplied by numeric skills and by other potential abilities of our mind.

The form of presentation of the number in question may be rather important and the circumstance that the number is constructed in such a way: firstly we multiple twelve 5' and to that number we add the product of ten

2's (this makes only 1024). The character of construction of that number will be surely used in one or another way.

Attending to school during the math lesson in order to factorize an expression which is the sum of two squares $x^2 + y^2$ (as it is in our case) we often apply the addition and subtraction of 2xy. Will try to apply it and we will see what would then happen. We get

$$5^{12} + 2^{10} = \left(5^6\right)^2 + \left(2^5\right)^2 =$$

$$= \left(5^6\right)^2 + 2\cdot 5^6 \cdot 2^5 + \left(2^5\right)^2 - 2\cdot 5^6 \cdot 2^5 =$$

$$= \left(5^6 + 2^5\right)^2 - \left(5^3 \cdot 2^3\right)^2 =$$

$$= \left(5^6 + 1000 + 2^5\right)\left(5^6 - 1000 + 2^5\right)$$

$$= 16\,657 \cdot 14\,657 = 244\,141\,649$$

CHAPTER XXIV.
ENERGETIC NUMBERS

Another short numerical composition is connected with so-called *energetic* numbers. We will now announce what kind of numbers they are.

Definition. **An integer will be called *energetic* if its digits taking them from the left to the right are increasing.**

We could state that these digits at least with that increasing "compensate" their "lessening influence" to the magnitude of the number. We understand perfectly well that positional numeration system is called positional because the influence of its digits to the magnitude of integer going from the right to the left is increasing.

The question we raise or the problem we regard is the following one:

Given *N* is an *energetic* number. What could be told about the sum of digits of the number *9N*?

From the first sight it appears that these sums may be spread rather widely. It is also as clear as a day that these sums are divisible by 9 or are multiplies of 9.

Let us take perhaps the simplest *energetic* number 12.

Now

$$12 \times 9 = 108 \text{ and } 1 + 0 + 8 = 9.$$

Take another energetic number such as, for instance

89.

Now

$$89 \times 9 = 801$$

and the sum of digits of this number is again 9.

Somehow it still doesn't happen to get something different from 9.

We will make several other attempts hoping to get something different from 9. In a case if we wouldn't succeed in getting something different from 9 we will be forced to start with proof that such sums are always 9 what is a bit strange because *energetic* numbers sometimes are not very small ones.

Take as the third example a modest number

1378.

Then

$$1378 \times 9 = 12\,402,$$

and the sum of digits of that number is

$$1 + 2 + 4 + 0 + 2 = 9 \text{ (9 again!).}$$

Before starting a proof let us regard the biggest possible *energetic* integer

123 456 789.

Let us see will it really the sum of digits of the nine times bigger number again be 9? We have

$$123\,456\,789 \times 9 = 1\,111\,111\,101$$

And this number as a number possessing nine 1's as the only non-zero digits in its decimal expression of course has 9 as the sum of its digits.

Now we can do nothing but try to prove that it will always be so. From the psychological point of view after these four occasional examples we practically already fast believe that it would be always the case. Otherwise we would stand under suspicion of selecting specially prepared examples and that was not so.

The subtlest part of our proof will consist from one fast imperceptible operation which in our case will consist of presenting the number 9A as a difference of number 10A and A and of fulfilling this subtraction in column.

If our *energetic* number is written as *KLMNOPRST* then the number 9A we will get after subtraction

$$\begin{array}{r} K\,L\,M\,N\,O\,P\,R\,S\,T\,0 \\ -\ \ K\,L\,M\,N\,O\,P\,R\,S\,T \\ \hline \end{array}$$

Now we will consequently write down all digits of that difference starting from the right and moving to the left. We start from units digit. That digit of units will be of course

$$10 - T.$$

Further on the tenth digit won't be $T - S$ as it could appear from the first sight but it would be

$$T - S - 1$$

because operating in units digit we've "borrowed" 1 from the tenth digit.

Moving further to the left we would get the following digits

$S - R,\ R - P,\ P - O,\ O - N,\ N - M,\ M - L,\ L - K$

(because it was no other "borrowings") and finally we state that the last digit is *K*.

Now the sum of all digits would be (we are starting now from the top digits) is

$$K + (L - K) + (M - L) + (N - M) + (O - N) + (P - O) + (R - P) + (S - R) + (T - S - 1) + (10 - T) = -1 + 10 = 9$$

(all numbers which are in our case denoted by capital letters simplify each other in a what is called "telescopically way". It remains only the sum of numbers -1 and 10 giving 9 as a total or whole sum.

We earnestly confess that our proof formally suits only for 9-digital *energetic* integers (there is the only such 9-ditigal *energetic* integer which we have already seen being *energetic*!).

Indeed the only such integer is 123 456 789.

Nevertheless after our proof the reader is already convinced that for another *energetic* integers the proof we demonstrated would work as well only the expressions of *energetic* integers would be shorter.

This problem of Muscovite origin is also discussed in [8].

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