The LAIMA series

Romualdas KAŠUBA

ONCE UPON A TIME I SAW A PUZZLE

PART II

R. Kašuba. Once upon a time I saw a puzzle. Part II.

Rīga: University of Latvia, 2008. – 67 pp.

The book demonstrates psychological aspects of problem solving on the basis of contest problems for junior students. Nevertheless, the approaches discussed are of value also for highest grades, for teachers, problem composers etc. The text can be used by all those who are preparing to research in mathematics and/ or to math contests.

The final version was prepared by Ms. Dace Bonka.

The book is printed with the support of the company RIX TECHNOLOGIES.

ISBN

Reģ. apl. No. 50003107501

Iespiests SIA „Mācību grāmata”, Raiņa bulv. 19, Rīgā, LV-1586, tel./fax. 7325322

"Let us take them in order.
The first is the taste…"

The Hunting of the Snark,
by Lewis CARROLL, Fit the Second

INSTEAD OF INTRODUCTION

Once upon a time there happened a day when the Lithuanian team-contest in mathematics was born. Speaking simply and prosaically, some mathematical event came into being. In that team-contest, five students, usually from the highest school grades, during 4 hours have to deal with 20 problems (you may notice that 5 times 4 is also 20 and admit it to be a remarkable fact).

That 1st version of that team-contest happened A.D.1986 and since then it is repeated year by year. With the time following problem occurred. Imagine that you are going to Vilnius, the capital city of Lithuania, to participate in that team-contest and I am your younger brother. I also would like to go to Vilnius with you. And the elder brother answers, that he has nothing against it. But in order to go to any capital or otherwise remarkable city it is better to possess some reason or pretext.

So the author having heard about that and wishing to help in all such and similar cases invented some pretext for involved younger brother to go to the capital city with *the elder brothers. He organized, or invented, the associated individual contest for younger brothers or for the younger grades.*

The organizer had also his own daughter, approximately of that age, only a bit younger, so his understanding on the willingness was even deeper.

That first individual contest for youngsters happened A. D. 1999 and the problems were proposed for younger sisters and brothers of grades 5, 6 and 7.

Two years later it was split into two subsections with different problems proposed for grades 5 and even 6 and another for the forms 7 and even 8.

This year already the 10th edition of that contest took place.

The author of these lines got a very honourable proposal to prepare the English version of these problems together with solutions.

The original intention was to include all of them in one volume. But rather because of some technical difficulties or otherwise because of shortage of time it was decided to split them into 4 parts..

And one more thing should necessarily be told and explained and even as well as possible. That is the adoption or harmonization. Otherwise, introducing characters to make the solution itself a part of their achievements.

A good part of the first Olympiad problems was formulated "without heroes". Then in order to make all that more attractive all the heroes occurred later, sometimes even against the will of the composer.

It must be told in a very clear way that practically all problems as such are taken from other sources and only after that they are adopted, reformulated or otherwise structured inventing some steps into which the question of the problem is divided also in order to make all that more attractive.

These attempts by the author were welcomed by quite a lot of involved persons: by students, teachers and colleagues. Taking that into account and following also some other reasons and advices I reformulated also all problems of the previous years.

Listening to all that it is understandable that I tried to present also the solutions as some kind of discussion between the persons involved and some imaginary Advisory Board.

The readers may have their judgment whether the author succeeded in achieving his goals. All remarks especially those critical ones would be extremely welcomed.

The author is thankful for the noble editors of the LAIMA series for constant inspiring of the author to do something.

I am very much indebted to the Father of LAIMA project Professor Benedikt JOHANNESSON, who is the constructive optimist from any serious point of view you only may invent or imagine.

I am also very much indebted to Professor Agnis ANDŽANS, who was the first reviewer of my first Lithuanian book. He inspired me to make also the translation of that me first book into English.

I am also very much indebted to Aivaras NOVIKAS for his eagerness to discuss all things with me every day. His constant linguistic advices were of great help and importance.

And of course also great are my thanks to Mrs. Dace BONKA, who has prepared already three of my English manuscripts and, as I hope, will have enough patience to prepare also the fourth one.

Romualdas Kašuba

CONTENTS

ABOUT LAIMA SERIES

In 1990 international team competition "Baltic Way" was organized for the first time. The competition gained its name from the mass action in August, 1989, when over a million of people stood hand by hand along the road Tallin - Riga - Vilnius, demonstrating their will for freedom.

Today "Baltic Way" has all the countries around the Baltic Sea (and also Iceland) as its participants. Inviting Iceland is a special case remembering that it was the first country all over the world, which officially recognized the independence of Lithuania, Latvia and Estonia in 1991.

The "Baltic Way" competition has given rise also to other mathematical activities. One of them is project LAIMA (Latvian - Icelandic Mathematics project). Its aim is to publish a series of books covering all essential topics in the area of mathematical competitions.

Mathematical olympiads today have become an important and essential part of education system. In some sense they provide high standards for teaching mathematics on advanced level. Many outstanding scientists are involved in problem composing for competitions. Therefore "olympiad curricula", considered all over the world, is a good reflection of important mathematical ideas at elementary level.

At our opinion there are relatively few basic ideas and relatively few important topics which cover almost all what international mathematical community has recognized as worth to be included regularly in the search and promoting of young talents. This (clearly subjective) opinion is reflected in the list of teaching aids which are to be prepared within LAIMA project.

Twenty five books have been published so far in Latvian. They are also available electronically at the web - page of Correspondence Mathematics School of the University of Latvia http://nms.lu.lv. As LAIMA is rather a process than a project there is no idea of final date; many of already published teaching aids are second and third versions and will be extended regularly.

Benedikt Johannesson, the President of Icelandic Society of mathematics, inspired LAIMA project in 1996. Being the co-author of many LAIMA publications, he was also the main sponsor of the project for many years.

This book is the 8th LAIMA publication in English and the 33rd in general.

FOURTH LITHUANIAN MATHEMATICAL OLIMPIAD FOR YOUNGSTERS (2002)

Grades 5 and even 6

1. THE LEAST POSSIBLE CHESHIRE CAT'S SHARE

Let us agree that if we are excluding some elements from a set – such things happen every day – then we'll say that we are hiring them to that Cheshire Cat, or that they are His share. In that natural terminology, if we have the sample

$$\{1, 2, 3, 4, 5, 6, 7, 8\}$$

and we make numbers 1, 3, 5, 6 and 7 to be Cheshire Cat's share, then, of course, only the numbers 2, 4 and 8 remain in that sample and the product of those remaining numbers

$$2 \cdot 4 \cdot 8 = 64$$

is a square (as well as the third and also even the sixth power of an integer – but we don't need it!) of an integer number.

Billy Boy wishes to establish some extreme fact, that is, he wishes to know precisely:

At least how many integers in that sample must we make the Cheshire Cat's share so that the product of the remaining numbers would be again a square of a natural number?

2. TOM AND JERRY IN THE LONG FURIOUS STATE

It appeared that Tom, while being furious – and this is not so seldom and rare – makes from any single piece of paper 8 smaller pieces, and Jerry even 12. With the time their furious state lasted for hours, so their exhausted neighbours confronted themselves with a series of simple questions. They became interested whether it is possible that after such a fit state they might get

(α) 60 (pieces starting) from 1?

(β) 61 from 1?

(γ) even exactly 2002 of them starting from 1?

3. DULCINEA AND HER PASSION FOR REMARKABLE NEIGHBORING DIFFERENCES

Dulcinea, whose love for arithmetic wasn't absolute, according to our understanding, once felt a passion for doing something that would be particular and in the same time would have some relation to that famous *this-is-and-always-will-be-right* area (which is only another name for math). So she resolutely decided to find out whether it is possible or it is not possible to write all the numbers

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10,$$

each of them once, around some circle so that the difference of every two neighboring numbers would be not less then 4?

What do you think – will famous Dulcinea succeed?

4. ESTABLISHING HOW BIG EACH OF FIVE HEROES IS

Once upon a time there lived in that world only 5 but very noble integer numbers.

Their magnitudes remained deeply hidden and top secret, but all their possible sums you could get adding three of them in every way you could ever imagine taken in a natural order were known to be exactly these:

10, 14, 15, 16, 17; 17, 18, 21, 22, 24.

In those days, when all number's life was without remarkable computational background, it appeared practically impossible to establish how big each of them was. But once Man Friday went by. He did it. And we might ask – why not you?

Grades 7 and 8

1. ANOTHER (SQUARE) MINIMAL POSSIBLE CHESHIRE CAT'S SHARE

Removing from the collection ۩ of first 8 even numbers

2, 4, 6, 8, 10, 12, 14, 16

the sub collection ۞ consisting from

4, 8, 10, 14 and 16

we might observe that the product of remaining integers

2, 6 and 12

or

$$2 \cdot 6 \cdot 12 = 12 \cdot 12 = 12^2$$

is a square. As you must have already mentioned, in such a case ۞ is also said to be *the (square) Cheshire Cat's share.*

What a minimal number of elements in collection ۞ will such *(square) Cheshire Cat's share* now contain?

2. THREE HEROES FISHING TWO PERFORMERS FROM ONE EQUALITY

Hare Dare and Wolf Rolf together with Fox M(ath)ox once found the sharpest equation

$$3xy - x - 2y = 8$$

in the forest of numbers under the oldest oak.

(A) Dare is eager to detect two integers x and y such that they both as one pair (x; y) would suit the equation they've found;

(B) In a case of success Rolf claims that he would take care for another such a pair (x; y) of two integers x and y – should he dare?

(C) M(ath)ox claims if both Dare and Wolf would be successful in their care then she also will take care and she'll also do and dare find the third successful pair;

(D) Dare and Rolf under the guidance of M(ath)ox intend to realize a joint project to find all such pairs (x; y) of two integers x and y. What do you think, how much of such pairs would there be?

3. ASTONISHING GEOMETRICAL ADVENTURE

Trapezium is any quadrangle two opposite sides of which are parallel and another two are not. The diagonal of isosceles trapezium divides it into two isosceles triangles. Is it then indeed possible to check out how big the angles of this trapezium are?

4. I'M DIFFERENT FROM YOU BUT TOGETHER WITH THE SKIN OF MINE I AM ALREADY AS PRECIOUS AS YOU WITH THE SKIN OF YOURS, OR ABOUT SOME MYSTERY OF PAIRS

After her adventures in Wonderland Alice believes in some natural mysteries. So she does not believe that it might happen that adding some natural number n with the sum of it digits $S(n)$ she could ever get the same number as taking another natural number m and adding it with the sum of its digits or with the number $S(m)$.

Once in dream, which was not all the dream, Alice had seen such two natural numbers n and m, which were different but, when added with the sums of their digits, become equal. Such integers that

$$n + S(n) = m + S(m)$$

are said to form a *mysteriously unified pair*.

Alice decided to pursue the following three aims:

(۩) to detect by her own at least one *mysteriously unified pair;*

(۞) to clear out whether there are natural numbers k, m, n , l such that

$$k = m + S(m) = n + S(n) = l + S(l);$$

(☺) to find out whether it is possible to detect some 23 such numbers instead of only three numbers k, m, n.

A. D. 2002, Grades 5 and even 6, problem 1. The smallest possible Cheshire Cat's share in the set {1; 2; 3; 4; 5; 6; 7; 8}. Thoughts related to the solution

We remind that the possible (square) Cheshire Cat's share in any finite number set is any subset or part of it after removing which the product of remaining numbers is a square of an integer.

We must answer the question about the *smallest* possible (square) share in the set of first 8 natural integers.

Clearly – that's our first observation – that 1, if present, must always remain if only we are taking care about minimal possible (square) Cheshire Cat's share.

Also it is obvious that if the product of all integers of a number set is already a square of an integer then it is possible that Cheshire Cat may be vanishing with nothing.

That wouldn't be so in our case because the product of all our 8 first initial numbers or the product

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8$$

is

$$1 \cdot 2 \cdot 3 \cdot (2 \cdot 2) \cdot 5 \cdot (2 \cdot 3) \cdot 7 \cdot (2 \cdot 2 \cdot 2),$$

which is the same as

$$(2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) \cdot (3 \cdot 3) \cdot 5 \cdot 7$$

and, according to our convenient common abbreviations, is equal to

$$2^7 \cdot 3^2 \cdot 5 \cdot 7 .$$

That number is in no way big. It is just

40 320.

It is not a square because if it was a square then all degrees in that expression were even – but in our case not all of them are! So we can help that either omitting some of them completely or lowering some of them.

Proceeding in that way we must, of course, omit 5 together with 7 and somehow to lower the power of 2 up to even number. For that we must omit at least at least 3 integers: the only possibility to lower the degree of 5 (and 7) is to omit 5 (and 7), that is, both of them (already this makes the minimal Cat's share consisting of not less than 2 numbers). In addition we naturally must also to lower the degree of two (and that will add the third element to any possible share).

So the minimal share will contain at least 3 elements. It remains to demonstrate that this is possible.

Indeed, removing only

$$2, \ 5 \ \text{and} \ 7$$

we clearly see the product of remaining integers being

$$2^6 \cdot 3^2$$

or

$$576 = 24 \cdot 24 = 24^2$$

.

The answer.

The least possible (square) Cheshire Cat's share contains 3 elements.

A. D. 2002, Grades 5 and even 6, problem 2. Tom and Jerry got furious for a long

When Tom is furious he makes in a second from 1 piece of paper 8 instead, and Jerry would make even 12 from that. Imagining that they would remain

furious long enough could they both, acting together, produce

60 pieces;

61 pieces;

2002 pieces

starting always from that single piece of paper?

Instead and around the solution

Dealing with arithmetical problems and other exciting matters not the first day and regarding these neighboring integers

60 and 61

we might naturally expect that highly probably one of them would be possible to get while other would not.

The all-first question then would be – why?

Dealing with these cases after achieving some insights and making progress we'll probably get some remarkable understanding about our chances concerning that case with 2002 pieces.

Simply observing how the things with furious Tom and Jerry are continuing we noticed immediately that after single action of Tom we would have a total increase of 7 pieces. That is as clear as a day because

1 "old" piece is vanishing while 8 "new" pieces are appearing so that their difference makes

$$8 - 1 = 7$$

and that's why we claim that 7 is the total increase a number of paper pieces.

In the case of Jerry the principle remains the same, only "the number of appearing pieces is remarkably higher":

1 "old" piece is vanishing while already 12 "new" pieces are appearing making that increase being

$$12 - 1 = 11$$

and so 11 is the higher increase.

Modeling that and continuing our scientific discussions we could say that it is the same as if they would take the number 1 and then Tom would go on adding 7 to what he has while Jerry even 11, and then the question would be whether consequently processing in that way they ever could get 60, 61 or even 2002 pieces.

We also may assume that they both remain furious – otherwise they would lose temper and stop adding their 7's and 11's.!

Let us see where they could land with their consequence in anger.

Still another breaking idea might appear. Imagine that it is possible to achieve, say, 60 pieces starting from the single piece or 60 from 1. We could quickly start imagining and easily assuming that their angels (they just as we all believe in angels) are dealing together but doing something rather opposite as they are:

They are reducing for 7 or 11 instead of adding like Tom and Jerry are. In addition they are starting from 60 and not from 1 like Tom and Jerry are. Nevertheless the task of angels seems to be easier and be also applied by Tom and Jerry – or they may start cooperating as usually people do with their angels!

So let us overwrite all this:

Tom and Jerry – being no more furious but already arithmetically engaged – start together with their angels from the number, say, 61. They may freely and in any consequence apply 2 procedures:

(Θ) Subtract 7;

or

(Ξ) Subtract 11.

The question is whether they are able to land at 1applying these 2 procedures.

There is not the slightest doubt that afterwards the task will be how to land at 1 changing the start number firstly to

60

and then rather boldly even to

2002!

We frankly say that the affair with landing at 1 when started by 61 went smoothly – take just a look and enjoy the whole.

61 → Θ → 54 → Θ → 47→ Θ → 40→ Θ →33 → Θ →
26 → Θ → 19→ Θ → 12 → Ξ → 1

and we arrived at 1 as we dreamed and needed.

We equally frankly say that it appeared already impossible – even with the strong interfere of angels – to land at 1 applying the same procedures when we are starting from 60 instead of former successful 61.

You understand what does it mean?

Even with the help of angels you cannot do such simple thing.

Then how will it be in the wicked case – 2002!

It is slightly thrilling even to start thinking about these endless possibilities and our desperate attempts to get through all that number industry.

Let us act preserving cool head and go on simply analyzing all this. It ought not to be so difficult and hopeless as it might seem. The science will help us.

We will start analyzing – why it indeed appeared impossible to get 1 starting from 60? Still we do remember how easy we landed at 1 starting from 61.

Let us assume the contrary and see what will happen. Let us assume that landing at 1 starting from 60 is possible.

Getting 1 from 60 means reducing of

$$60 - 1 = 59$$

pieces after applying some number ☼ of operations of

Θ type ("*subtract* 7")

together with some another number ☺ of operation of

(Ξ) type ("*subtract* 11").

If it is possible to get 1 starting from 60 then it must naturally be that

7☼ + 11☺ = 59

Now always remember that ☺ and ☼ mean exactly how many times Tom, Jerry, their angels with all of us as observers applied these operations. So ☺ and ☼ might be only 1, 2, 3, …, or even possible 0 – that would indicate that the given procedure wasn't applied or otherwise involved.

But collecting those needed 59 using 7's and 11's or their multiplies we have only very few possibilities or exactly these ones:

Imagine ☺ would be 0; then 7☼ = 59 but this is impossible in integers;

Imagine ☺ would be 1; then 7☼ = 48 but this is impossible in integers;

Imagine ☺ would be 2; then 7☼ = 37 but this is impossible in integers;

Imagine ☺ would be 3; then 7☼ = 26 but this is impossible in integers;

Imagine ☺ would be 4; then 7☼ = 15 but this is impossible in integers;

Imagine ☺ would be 5; then 7☼ = 4 but this is impossible in integers;

And "☺ = 6" would already mean

$$7☼ = – 7$$

or

$$☼ = –1$$

(☼ is an integer – before it never was – but it's of little help because it's negative!)

and so that is "more than impossible" – could you ever imagine yourself doing something "minus once"?

So there are no possibilities for the equation

$$7☼ + 11☺ = 59$$

to be solved in non-negative integers ☺ and ☼.

Still it could be repeated that in the same time it is possible to solve in positive integers the "neighboring" and so similar equation

$$7☼ + 11☺ = 60$$

because as we all remember detecting that

$$(☼; ☺) = (7; 1)$$

is the sure solution because nobody could deny the fact that:

$$7 \cdot 7 + 11 \cdot 1 = 60!$$

Now the main case is how to land at 1 starting from 2002? That would mean exactly the ability to solve in non-negative integers the equation

$$7☼ + 11☺ = 2001.$$

We have seen in what way it was done. It was indeed so simple. No theory at all was either applied or needed. All what we'll hear in the coming process of solving would be the following fact, which is clear without any modification, translation or other explanation:

If in the correct equality

$$a + b = c$$

2 of 3 performers, those two of

$$a, b, c,$$

are indeed integers then so is also the third performer! We will refer to that fact by saying

"*2 from 3 well, all of them well*".

It was followed by the following words:

$$7☼ + 11☺ = 2001 \qquad \textbf{(1)}$$

implies

$$7☼ = 2001 - 11☺$$

or after reordering

$$7☼ = 2001 - 11☺ = 2002 - 14☺ - 1 + 3☺ =$$
$$= 7(286 - 2☺) + (3☺ - 1).$$

After division by 7 we get

$$☼ = (286 - 2☺) + (3☺ - 1)/7 \qquad \textbf{(2)}$$

Now we apply for the first time that "*2 from 3 well, all of them well*" or, more prosaically, that:

$$☼ \text{ and } 286 - 2☺$$

being integers,

$$(3☺ - 1)/7$$

also is forced to be one! Giving to that third integer the special new name ◙ or setting

$$◙ = (3☺ - 1)/7$$

we'll have

$$7◙ = 3☺ - 1$$

or

$$3☺ = 7◙ + 1$$

Again reordering

$$3☺ = 6◙ + ◙ + 1$$

we enjoy for the second time “*2 from 3 well, all of them well*”!

$$☺ = 2 ◙ + (◙ + 1)/3 \quad \textbf{(3)}$$

Again the same expressed in everyday words sounds as follows:

☺ and 2 ◙

being integers, also

(◙ + 1)/3

is forced be an integer. Long may live under the name ♣ after setting

$$♣ = (◙ + 1)/3 \quad \textbf{(4)}$$

meaning that

3 ♣ = ◙ + 1

or

◙ = 3 ♣ − 1

Now we are coming back expressing these initial heroes’ ☼ and ☺ also in terms of ♣.

From **(3)** and **(4)** we conclude that

☺ = 2 ◙ + (◙ + 1)/3 = 2(3 ♣ − 1) + ♣ = 7 ♣ − 2

and using **(2)**

☼ = 286 − 2 ☺ + (3 ☺ – 1)/7 = 286 − 2 ☺ + ◙ =
= 286 – 2(7 ♣ − 2) + 3 ♣ − 1 = 289 – 11♣

we finally enjoy that

☼ = 289 – 11♣
☺ = 7 ♣ − 2

Let us check that we did everything correctly. Indeed, plunging these expressions into initial equation **(1)** or into

7☼ + 11 ☺ = 2001

we’ll get

7(289 – 11♣) + 11(7 ♣ − 2) = 2001

or

$$2023 - 77♣ + 77♣ - 22 = 2001,$$

leading us to the desired tautology in form of

$$2001 = 2001.$$

Instead of final remarks we'll say what follows.

What do we achieve and establishe? We do achieve that the series

$$(289 - 11♣;\ 7♣ - 2)$$

forms a solution series for the equation

$$7☼ + 11☺ = 2001$$

independently from what value ♣ takes!

And that ♣ should be an integer from the very infinite set

$$\{\ldots -3; -2; -1; 0; 1; 2; 3;\ldots\}.$$

Of course, those two celebrated expressions should be non-negative.

So taking ♣ for instance 1, we will get a particular solution

$$(289 - 11 \cdot 1;\ 7 \cdot 1 - 2)$$

or

$$(278; 5)$$

which means that there are 278 single deeds of Tom (8 from 1) and only 5 of Jerry (12 from 1).

And by the way that isn't the only possibility for the brothers dealing together to produce 2002 from 1.

The answer.

It is possible to get 61 or even 2002 pieces of paper starting from a single piece but it is impossible to get 60 pieces starting from that single piece.

A.D. 2002, grades 5 and even 6, problem 3. First math period of Dulcinea (together with the first success – we are not going to hide the truth!)

In the preliminary preamble it came out that Dulcinea wasn't a fan of math. Nevertheless, being resolute she did it.

We would like to remind you that she was asked to find out whether it is possible to write around the circle all integers 1 to 10 with pairwise differences of any two neighbors not less then 4.

You may wonder, you may smile, but you can not deny that at that point she was smart.

Look here! She did it! She did it!

Below we present her answer.

5
10 9
1 4
6 8
2 3
7

The correct answer – yes, it is possible, and Dulcinea managed to get all integers 1 to 10 around the circle in the way that a difference of any pair of neighbors is 4 or more. Since the right answer is presented, for some short moment all essential questions are exhausted.

A.D. 2002, grades 5 and even 6, problem 4. 5 noble numbers together with their magnitudes. Around the solutions with useful considerations and some possibly not bad reflections

Let us tell you what that diligent Man Friday did. You must know that living alone for years he was used to speak loudly what he was thinking about. So he just started thinking, that is, speaking load: just imagine those numbers be

a, b, c, d and e.

By that Friday already baptized them.

Afterwards he mentioned that they all are participating in this just in the same way – independently from how large they are or might be.

Democracy online in general and by us already in action!

We could speak also about some full symmetry.

Further on the following remark was to be announced and even written down:

Each of them, say a, is involved 6 times or is acting in 6 sums:

$$a+b+c,$$

$$a+b+d,$$

$$a+b+e,$$

$$a+c+d,$$

$$a+c+e,$$

$$a+d+e$$

just as all others b, c d and e of them are.

Then he went ordering them by magnitude

$$a<b<c<d<e$$

Remembering that each of them is *the six time participator* and finding the sum of all those possible 3-sums we would get 6 times as big sum as all of them together are, or

$$6(a+b+c+d+e),$$

and this 6 times as big *sum of sums* would and must be

$$6(a+b+c+d+e) =$$

$$= 10+14+15+16+17+17+18+21+22+24 = 174$$

or

$$a+b+c+d+e = 29$$

Afterwards the diligent Man went on and along for a while humming and once and again repeating that only refrain:

"All of a, b, c, d and e are in fact different, are different, are different because otherwise I would enjoy (at least) some 3 equal sums among these 10 provided and not just only 2 as it happened in my case"!

Further there were some essential but already exceptionally technical details. Clearly the smallest from all these 3-sums or

$$10$$

will, of course, represent the sum

$$a + b + c$$

of three smallest integers a, b, c.

Similarly the biggest sum or

$$24$$

will, of course, represent the sum

$$c + d + e$$

of those three biggest numbers c, d, e.

Then the sum of these both two extreme sums or

$$10 + 24 = 34$$

is, on the other hand, the sum

$$a + b + 2c + d + e$$

and keeping in mind that

$$a + b + c + d + e = 29$$

we'll state that the middle number c is exactly 5.

So if 5 stand exactly in the middle of those 5 – so let him stand! Then we are waiting for some kind of breaking idea and that breaking idea appears in so simple form of an average-looking statement:

The second smallest of all these 3-sums is always the sum

$$a + b + d,$$

which in our case makes

$$14$$

and respectively the second largest of all these possible 3-sums is always

$$b + d + e$$

or

$$22$$

in our case.

But

$$a + b + c = 10$$

together with

$$c = 5$$

implies that

$$a + b \text{ is also } 5.$$

Together with the fact

$$a + b + d \text{ being } 14$$

this means

$$d = 9.$$

Remembering again that the biggest of 3-sums is 24 we might extract the sure fact that

$$e = 10.$$

Now the second biggest 3-sum or

$$b + d + e \text{ being } 22$$

together with

$$d + e = 19$$

(*because* $c + d + e = 24$ *with* c *being* 5) *implies that*

$$b = 3.$$

Together with

$$a + b = 5$$

and remembering that b is 3 as it was just being stated implies

$$a = 2.$$

Now Friday has found all numbers

2, 3, 5, 9 and 10.

He completed that process by writing thoroughly down all these possible 3-sums again:

$$2+3+5,\ 2+3+9,\ 2+3+10,\ 2+5+9,\ 2+5+10,$$
$$3+5+9,\ 3+5+10,\ 2+9+10,\ 3+9+10,\ 5+9+10.$$

The answer.

Man Friday found those 5 top secret and completely hidden integers being 2, 3, 5, 9 and 10.

A.D. 2002, Grades 7 and even 8, problem 1. Establishing the minimal square that Cheshire Cat's might share in the set of first 8 even possitive integers

You might have already noticed – surely you did – if product of all numbers in the given set is already the square of integer then it is also possible that the Cat is vanishing with nothing or, scientifically speaking, his share might be also empty.

In our case writing all these 8 numbers in their prime form and multiplying them we would get

$$2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14 \cdot 16 =$$

$$2\cdot(2\cdot 2)\cdot(2\cdot 3)\cdot(2\cdot 2\cdot 2)\cdot(2\cdot 5)\cdot(2\cdot 2\cdot 3)\cdot(2\cdot 7)\cdot(2\cdot 2\cdot 2\cdot 2)=$$
$$=2^{15}\cdot 3^{2}\cdot 5\cdot 7$$

Trying to omit some numbers in the initial collection ۩ in order that the product of all remaining numbers would be the square we clearly understand that then we ought to say farewell to 5 and 7 – in other words in the initial collection we are forced to omit 10 and 14. Omitting them we will also lower by two 2's the degree of 2 in the product of off all numbers in ۩ . But this means that the degree of 2 in the whole product of remaining numbers remains odd. That could be changed to even by taking away, for instance, the first number 2 from the initial collection.

This clearly indicates and makes us surely to believe that any possible (square) Cheshire Cat's share ۞ must contain at least three numbers and also demonstrate that three are possible to achieve.

In such minimal case that share ۞ might, for instance, consist from

2, 10 and 14.

That's not the only possibility because instead of taking 2 to that minimal share ۞ we could take, for instance, 8.

So then

8, 10 and 14

would form another minimal (square) Cheshire Cat's share ۞.

The answer.

The minimal Cheshire Cat's share contains 3 numbers.

A.D. 2002, Grades 7 and even 8, problem 2. Around and instead of solution – some words of report

Hare Dare promised one integer pair of suitable integers for equation

$$3xy - x - 2y = 8,$$

Wolf then promised another and Fox afterwards even the third pair, and after that they all started acting as united team, promising and bringing to us all integer pairs, which are good for that equation.

Spoke the hare:

- Let us expel from that equation absolutely all summands where x can be seen – independently alone or as a multiple – or let's simply forget about x – as if it never existed.

- It's high time for that, - agreed wolf, – but what's then?

After it was done they all examined the rest of that remained or

$$-2y = 8$$

bringing immediately

$$y = -4.$$

- So dear Hare, what a pair you've got?- asked Fox Mathox.

- I've got 4, – boldly added Dare.

- But 4 is not a pair, it is, as you may possibly understand, the single number.

And both of them – hare and wolf – started to stare for the first moment without the slightest understanding, what a pair they've got totally forgetting *x* in that original equation.

Their contemplation could probably last for hours but that silence was suddenly interrupted by the singing voice of fox Mathox:

- You will understand some days later that completely forgetting x means that you are taking that

$$x = 0.$$

- And why this is so?

- You might imagine on your own or simply check that a pair

$$(0; -4)$$

is indeed suitable for our equation.

They carefully checked it, and the pair was indeed a suitable pair, and so their admiration for the fox was rising up to the skies.

Now the hare Dare told:

So now that pair (0; –4) being the first pair is my solution.

And this accepted by all of them.

Needless to say that afterwards wolf Rolf did the same with the only difference that now he was completely ignoring y. This leaded our company to that what remained when y was completely erased from the initial equation or

$$-x = 8$$

giving not only that

$$x = -8$$

but remembering the fox's observation that completely erasing y means simply to set out that y is 0 or that the pair

$$(-8; 0)$$

is another solution. It was again accepted by all of them as the solution for the wolf.

Now both of them – hare with wolf – wondered whether the fox will again be able to invent something new in the area how to find something of importance and demonstrate them how to find some solution doing practically nothing. And their hopes were not in vain because the fox declared:

- It is also not bad, erasing x, to let other terms to exist just as they were, and look again what will happen then!

Now the wolf proved itself to be cool because he asked:

- And what'll remain when we, speaking in your terms, will erase x in expression

$$-x$$

and only x, letting all other terms exist further just as they were?

– What will remain then is for me as clear as the day, – laughed the fox.

– And what will remain then, - hare was also still out of clear understanding what's going on.

– Think bright, - patiently repeated the fox, – apply additional capacities, for instance, you may imagine that instead of

$$-x$$

you have to deal with

$$(-1)\,x.$$

Then it will be no difficulties erasing only x but not the terms, which are multiples together with it.

It remains then

$$3y-1-2y=8$$

or

$$y=9.$$

- And what's now with x? Does it mean again that x is understood to be 0? – asked wolf.

- Of course, not 0, because our proceedings now are quite different as they were just before, not at all 0, not at all 0…

– Not 0, but exactly1, – added hare Dare.

Exactly, - confirmed fox.

So they got already the third solution at a time, or

$$(1; 9)$$

and later also Ralf found also the fourth pair erasing only y and never what's standing together as multiples with y:

$$3x - x - 2 = 8,$$

$$2x = 10$$

giving

$$x = 5$$

or the pair

$$(5; 1).$$

In advisory board, hearing all that philosophy of these bright animals, we were already eagerly waiting also for some general methods understanding that all these guessing methods, precocious as they were, do not guarantee all solutions. We waited for these general methods being also ready to help them if necessary. But there was no great need for that because fox Mathox appeared technologically, as expected, rather skilled, and under her guidance all these matters were going on more or less in the manner, as it will be told below. We omit some smallest details and we'll tell you only the main things which were taking place:

$$3xy - x = 2y + 8,$$

$$x(3y - 1) = 2y + 8,$$

$$x = \frac{2y+8}{3y-1}.$$

Now the breaking consideration: assuming x and y are integers then

$$3x$$

is also an integer. But then

$$3x = 3 \cdot \frac{2y+8}{3y-1} = \frac{6y+24}{3y-1} = \frac{2(3y-1)+26}{3y-1} = 2 + \frac{26}{3y-1}.$$

Now $3x$ is an integer, so is also

$$\frac{26}{3y-1}$$

as well.

But $\frac{26}{3y-1}$ is an integer exactly when $3y-1$ is one of the divisors of 26, these divisors being

$$\{-26, -13, -2, -1, 1, 2, 13, 26\}.$$

Then $3y$ is one from numbers

$$\{-25, -12, -1, 0, 2, 3, 14, 27\},$$

and y is an integer only in 4 cases or when

$$y = -4, \quad y = 0, \quad y = 1 \quad \text{and} \quad y = 9,$$

and then

$$x = 0, \quad x = -8, \quad x = 5 \quad \text{and} \quad x = 1.$$

Observation. As we see we've guessed all these solutions before. But the general solving of equations is in general absolutely necessary, because even the most excellent guessing might lose some partial or even more general cases. Of course, solving quadratic equation when it is for sure known and granted that it has no more that 2 solutions, after indicating two numbers, which are indeed solutions, we may already stop solving. Or in other similar cases, if someone guarantees that the equation has not more than 55 solutions and we‘ve found them already, then, of course, we can also stop solving. In other cases when we do not know exactly how many solutions are there, then the “general solving” is absolutely necessary – otherwise we might lose something, so making our solution incomplete.

Second possible method of solution might be the following, and it is clearly related to the first method.

Multiplying that equation

$$3xy - x - 2y = 8$$

by 3 we can overwrite it as

$$9xy - 3x - 6y - 24 = 0$$

or

$$(3x - 2)(3y - 1) = 26\,.$$

Now

$$26 = (-26)(-1) = (-13)(-2) = (-2)(-13) = (-1)(-26) =$$

$$= 1 \cdot 26 = 2 \cdot 13 = 13 \cdot 2 = 26 \cdot 1$$

so we’ll get 8 similar systems

$$\begin{cases} 3x - 2 = \text{some divisor of } 26\,, \\ 3y - 1 = \text{complementary divisor of } 26. \end{cases}$$

From those 8 systems of equations 4 systems lead to integer solutions, and so we get again these 4 well known solutions: (–8; 0), (0; –4), (1; 9) and (5; 1).

The answer: (–8; 0), (0; –4), (1; 9) and (5; 1).

A.D. 2002, Grades 7 and even 8, problem 3. Amazing fact about cutting of some rare kind of isosceles trapezium into two isosceles triangles by drawing the diagonal. Considerations leading to better understanding that then we might detect how big these angles of that trapezium were before cutting.

We might remind to all whom it may concern that the trapezium is a quadrilateral having exactly one pair of mutually parallel sides.

Also in an isosceles trapezoid, the angles at the same base are equal, and so are the lateral sides.

To solve the given problem it is absolutely enough to know the following facts.

1. Intersecting two parallel lines by the third line the alternate angles are equal and, conversely, if these alternate angles are equal then the lines are parallel.

2. The two angles adjacent to the third side (base) of an isosceles triangle are equal and, conversely, if these adjacent angles are equal then that triangle is an isosceles triangle.

3. The sum of the angles of a triangle is 180°.

Now looking to the picture below and seeing these two isosceles triangles, which we've got providing the diagonal *BD,* we'll try to make some practically obvious and not very complicated conclusions.

The fact that

$$BC = CD$$

implies, as mentioned,

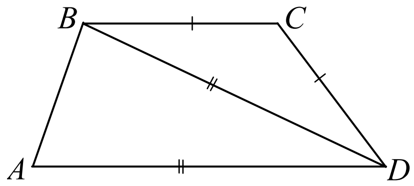
$$\angle CBD = \angle CDB$$

and taking into account that

$$BD = AD$$

we get that

$$\angle DBA = \angle DAB .$$



If $\angle CDB = \alpha$ we get also

$$\angle CBD = \angle BDA = \alpha$$

(alternate angles).

Then in the triangle ΔBDA for two equal angles at the base AB remain $180^\circ - \alpha$ degrees, or for one angle DBA exactly $90^\circ - \dfrac{\alpha}{2}$ degrees.

In the isosceles trapezium the sum of two opposite angles is 180° so that

$$\angle CBA + \angle CDA = 180^\circ ,$$

leading to

$$\angle CBA = \alpha + 90^\circ - \frac{\alpha}{2} ,$$

$$\angle CDA = \angle CDB + \angle ADB = \alpha + \alpha = 2\alpha ,$$

and

$$\alpha + 90^\circ - \frac{\alpha}{2} + 2\alpha = 180^\circ ,$$

giving

$$\frac{5}{2}\alpha = 90^\circ$$

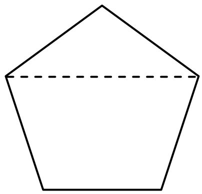
and finally

$$\alpha = 36^\circ .$$

That means that the angles of the trapezium are

$$2 \cdot 36^\circ = 72^\circ \text{ and } 180^\circ - 72^\circ = 108^\circ.$$

Final remark. One may notice that exactly such trapezium as we've just discussed might appear and appears when we are cutting off one angle of regular pentagon by the line passing through the two neighbouring vertices. In any regular pentagon, just as in every pentagon, which is "made" of 3 triangles, the sum of all 5 internal angles is $3 \cdot 180^\circ = 540^\circ$, and for one angle in the regular pentagon we have then $540^\circ : 5 = 108^\circ$.



The answer.

The angles of our isosceles trapezium are 72° and 108°.

Comment of the editors. Of course, in a full solution, it must be also proved that the equal sides of isosceles triangles **are exactly those** which are considered as such in the given solution.

A.D. 2002, Grades 7 and even 8, problem 4. Around the solution and 3 aims of Alice concerning the mystically unified pairs and what's happened

We remind that Alice was eager to find by her own two different positive integers n and m such that

$$n + S(n) = m + S(m).$$

Such pairs were referred to be *mystically unified*.

It ought to be reported that Alice in 5 minutes found out one such a pair of integers

92 and 101.

Indeed

$$92 + (9 + 2) = 101 + (1 + 0 + 1) = 103,$$

as required.

Honestly speaking so far we have no reports about other successes and achievements of Alice. For instance, her second step was to detect three and afterwards even more such integers giving the same result when summed up with its digits.

The (partial) answer.

Integers 92 and 101 constitute a pair of mystically unified integers.

THE 5TH LITHUANIAN INDIVIDUAL MATHEMATICAL SCHOOL OLYMPIAD FOR YOUNGSTERS (2003)

Grades 5 and even 6

1. BILLY BOY IS DEALING WITH PALINDROMS

Billy Boy together with us is aware that an integer number is called palindrome if it remains the same independently how we are reading it: from the right to the left or conversely – from the left to the right (e.g. 7227 is a palindrome number).

(☺) Today Billy Boy wishes to find a palindrome number ending by 27, divisible by 27, and his strong wish also is that the sum of the digits of it is again equal to 27;

() Tomorrow his plans will already include determination of at least 3 such numbers;

() Finally and naturally on Saturday we will find the smallest such number.

What a number would appear as the smallest one among all of such numbers?

2. ARGENTINIAN GAUCHO NAMED BRUNO IS STILL IMPROVING THAT WORLD

In each entry of the 4×4 square a "+" sign is written.

+	+	+	+
+	+	+	+
+	+	+	+
+	+	+	+

An Argentinean gaucho named Bruno (with all of us in the status of his honorable advisers) is intended to implement the most challenging project ever mentioned. In one move it is permitted to changes all signs in any 2×2 sub-square into the opposite ones. From that initial square with all 16 + signs in all entries making several such moves as it was just described he dares to achieve such a sign configuration in which plus and minus signs are arranged in the chessboard manner. Is he a little bit crazy with that entertainment?

+	–	+	–
–	+	–	+
+	–	+	–
–	+	–	+

3. INDIANA JONES SETTING DISTANCES 1, 2, 3, 4, 5 AND 6

In the period of Christmas the honorable Indiana Jones once in the dream eagerly wanted to solve such an intellectual puzzle:

Is it possible to detect in the plane 4 such points so that all 6 possible distances between those points were 1 cm, 2 cm, 3 cm, 4 cm, 5 cm and 6 cm?

4. THAT WASN'T THE END OF SWEET MOLLY MELONE

Sweetest Molly Melone in Dublin, the fair city, found once the log of the length of 100 m, which has been cut up into 30 pieces each of those being either 3 or 4 meters long.

Now she is piloting the recycler project, which seeks to arrange that all these pieces would be cut up to the pieces of 1 meter long each.

Can you help her ever establish how many cuttings will be necessary for that?

5. AND THAT WERE AGAIN TOM AND JERRY, THEY ACTED IN TURN FOR A WHILE

On the blackboard the numbers 1, 3, 4, 6, 8, 9, 11, 12 and 16 are written. Serious-minded again, Tom and Jerry in turn, one after another, are crossing out four numbers each. Tom starts first, so Jerry is the second. The historians stated that sum of all 4 numbers crossed out by Tom appeared to be strictly 3 times as big as the sum of numbers, which were crossed out by Jerry.

The crucial question, which still remained open and permanently discussed, is: whether or not it is possible to establish what number finally remained on the blackboard?

Grades 7 and even 8

1. GRANNY'S PERMANENT ACTIVITIES

Granny-25 is the great fan of such numbers which could be written only and exceptionally by 2's and 5's (not necessarily both). It must only be strongly mentioned and repeated that in whole Granny's family the only one thing is forbidden – in no number two 2's can ever be neighbours (no number can contain two consecutive 2's).

Numbers as mentioned are known *as a two-five-two numbers* in their environment.

(♣) In grade 2 Granny made a list of all 5-digital *two-five-two numbers.* How many numbers were on his list?

(♠) In grade 5 Granny made another list of all 10-digital *two-five-two numbers.* How many numbers were now on his list?

2. ALI BABA SPLITS THE TRIANGLE-40

The recent intellectual deeds of the famous broker Ali Baba included his famous split in the triangle-40. Here is that problem.

On the median BM of the triangle ABC such a point S is chosen that BS = 3SM. A line passing through the points A and S intersects the side BC at a point N.

Find the area of the quadrangle CMSN if the area of the initial triangle ABC is 40.

3. DULCINEA AND SUPERLONG NUMBERS

Dulcinea is very critical as to the dreams of Don Quixote to meet one day or another on the country side road such a long long number, which ends by 2003, is divisible by 2003 and has the sum of all digits also 2003.

4. CANTERBERRY FRACTIONS

Johnny the Optimist wrote 11 fractions using all natural numbers from 1 to 22 exactly once – either as numerator or as denominator.

What is the largest number of these fractions, which are integer numbers?

SOLUTIONS WITH HINTS AND POSSIBLE ADVISES

A. D. 2003, Grades 5 and even 6, problem 1. Billy Boy and his palindromes, divisible by 27: first one example, then some three of that kind and finally we will find, of course, the smallest one among all these palindromes divisible by 27

1. Ad (A) or how to find at least one of such numbers.

Start from the funny suspicion that already the palindrome mentioned in the condition might probably do.

That one mentioned in the condition, or

$$7227$$

clearly wouldn't and the reason is more than prosaic – the sum of it digits is too small; it's obviously only

$$7 + 2 + 2 + 7$$

or only 18 – too small.

This wouldn’t be the only reason – another reason is that the number 7227 is also not divisible by 27: in order to believe and state it for sure it is enough to divide it firstly not by 27 but by 9 only and then to examine whether the result will still be divisible by 3.

It is indeed not so because pretending to make long division we get

$$
\begin{array}{r|l}
7227 & 9 \\
\underline{72} & 803 \\
27 & \\
\underline{27} & \\
0 &
\end{array}
$$

and 803 isn’t divisible by 3 because then the number 800 would be divisible by 3. But 800 isn’t divisible by 3.

But this number has clear merits – these merits being the ending by 27 and being palindrome so we will keep on it for a while – trying, of course, to modify it.

The wish to keep the number palindromic and to increase its sum of digits in order to make it 27 may lead to inserting of 9 in the middle of the number or to the number

$$72927.$$

This, can you it ever imagine, already works, because the new enlarged number after division by 9 is still divisible by 3. Indeed we get

```
7 2 9 2 7   |9
7 2          8 1 0 3
  9
  9
    2 7
    2 7
      0
```

And 8103 is divisible by 3.

So the number 72927 is the answer for (A), because it is an example of such a number – palindrome, divisible by 27, ending by 27 and with sum of digits also 27.

But this is also an answer for part C because our construction demonstrates that the number

72927

is also the smallest among them all.

There remains the Part B asking for 3 examples. One of them we have; it remains to find another two.

Many ideas are possible to apply in so many places – and one of such ideas is to insert zeroes in suitable places in order not to loose the needed properties.

Having that in mind it is easy to see that the following two examples might be

7209027

and then

720090027

and so on.

Answer for (B): e.g. 72927, 7209027, 720090027,….

A. D. 2003, Grades 5 and even 6, problem 2.

You will probably laugh in a desperate way, but in the life rather often simple things are not realizable, but unbelievable things very often are!

Just take a look. Till we, advisers, gathered for the special discussions whether it is possible and discussed it thoroughly, the Argentinean did it! We have just seen it! It runs!

Unbelievable!

+	+	+	+
+	+	+	+
+	+	+	+
+	+	+	+

The very beginning is more than clear. In four moves – dividing that 4 x 4 square into 4 usual 2 x 2 sub squares Bruno can change all + signs into the opposite – signs:

–	–	–	–
–	–	–	–
–	–	–	–
–	–	–	–

Now we've noticed that Bruno intended and changed all signs in the inner 2×2 square:

–	–	–	–
–	+	+	–
–	+	+	–
–	–	–	–

Now Bruno changes slightly his manners and firstly changes the signs in the left-most 2×2 square located in the second and third row:

–	–	–	–
+	–	+	–
+	–	+	–
–	–	–	–

Using the usual chess notation changes happened in the "a2b2b3a3" square.

Now the changes are going to take place in the right-most 2×2 square located also in the second and third row or "c2d2d3c3" square:

–	–	–	–
+	–	–	+
+	–	–	+
–	–	–	–

Now the changes are going to happen firstly in the middle 2×2 square in the third and fourth row or in "b3c3c4b4" square and then in the 2×2 square, which is strictly below that square, or in "b1c1c2b2" square.

We get then

–	+	+	–
+	+	+	+
+	+	+	+
–	+	+	–

Actually now he is again changing signs in the "internal" 2×2 square "b2c2c3b3":

–	+	+	–
+	–	–	+
+	–	–	+
–	+	+	–

It could be mentioned not without some satisfaction that right now he is dealing with the highest top-left 2×2 square “a3b3b4a4” and so our eyes have to enjoy the following mutation:

+	–	+	–
–	+	–	+
+	–	–	+
–	+	+	–

And now Bruno is busy in the lowest 2×2 square “c1d1d2c2”, which is the most-left.

And that’s all – we are done!

+	–	+	–
–	+	–	+
+	–	+	–
–	+	–	+

The answer.

Yes, this is possible as it was shown above.

A. D. 2003, Grades 5 and even 6, problem 3. 4 Points of Indiana Jones all located in one and the same plane with all possible mutual distances between then filling out the whole set {1; 2; 3; 4; 5; 6}. It seems to be the hard nut to knack

But what is difficult in bright waters sometimes is easier in the narrow ones – or on the line it is easier than in whole plane.

You will be given an example.

This is a line.

On that line we will mark four points: firstly A in any place we wish! Then, moving along that line and never changing the direction, we will mark another point B so that the distance between A and B would be exactly **2 cm**. Mowing further we will mark the point C so that the distance between B and C would be **3 cm** (making the distance between A and C already **5 cm**) Moving still in the same direction as we were moving all the time we finally mark the last point D so that the distance between C and D would be only **1 cm.**

All these markings are possible even if it would be raining cats and dogs. (Then we could use them for marking!) The last marking of the point D makes the distance between A and D be **6 cm** and between B and D – **4 cm**.

—A ———————B ————————————C ———D ——

2 cm **3 cm** **1 cm**

So everything is arranged – a suitable example is the best proof!

A. D. 2003, Grades 5 and 6, problem 4.
Molly Melone and the 1og which once was a 100 meters long but in moment when it was presented to the famous Molly it was already being cut in 30 pieces, each piece either 3 or 4 meters long. The realities of life were such that of all these pieces 100 one-meter long pieces ought to be made. Molly, sweetest Molly, was expected to determine how many cuts would be necessary for that. We still believe that in extreme case some of them could be taken away using her wheel barrow.

Solution

Frankly speaking, going to observe what might happen here we didn’t await anything very complicated –

the problem as such is not at all as hard as we have sometimes seen. We were prepared to explain it to Molly either using two unknown magnitudes or, otherwise, using non-formal arithmetical considerations applying some elements of barter exchange in sense of:

Let us assume or imagine that long long ago all these 30 pieces were 3 meters long – each of them…, then, so what's then…. ?

We even made some short notices for each of these two cases:

1 case (standard –normal solution with ♠ and ♣):

Suppose that there are

$$♠$$

pieces which are 3 meters long (so the total length of all where 3♠) together with the

$$♣$$

4 meters long pieces (having total length 4♣).

It was told and repeated that there were 30 such pieces in total (♠ and ♣): in our notation this clearly indicates that

$$♠ + ♣ = 30$$

And once not very long ago all these 30 pieces were one apiece so that it could easily overwritten as

$$3♠ + 4♣ =100$$

and we are dealing with usual system of equations

$$♠ + \quad ♣ = 30$$

$$3♠ + 4♣ =100,$$

which admits an easily understandable way of solution eliminating one of the heroes of that system.

If we will thrice the first equation and then subtract it from the second equation we will get

$$♣ =10,$$

Which means that there are exactly ten pieces of length 4, subsequently there are twenty 3 meters long pieces (♠ = 20).

It remains to state that taking 3 meter piece we would need 2 cuts in order to get uniform 1 meter long pieces as is planned, and similarly we would need 3 cuts to get 1 meter long pieces from the 4 long stick.

For all in all we would need

$$20 \cdot 2 + 10 \cdot 3 = 70$$

cuts.

It is good and honorable but in the same time not too special or somehow complicated.

Paraphrasing in that page it could be added:

The standard calculation in standard book.

We have prepared also an arithmetical solution with some elements of non-standard imagination.

Let us imagine for a while that our sticks, sorry, pieces are all 3 meter long.

Putting them together we wouldn't get 100 meters as it originally was, but only

$$30 \cdot 3 = 90$$

or

$$100 - 90 = 10$$

"too little". So there will be not only 3 meter pieces, we'll have also those of 4 meters.

In order to improve or change the situation let's organize for our sweet MOLLY Melone the barter exchange. Using all ammunition we could say – borrow her wheel barrow and through "streets broad and narrow" take one 3 meters piece away and take instead of it another one – that of 4 meters long, and let it being transported with the same wheel barrow.

What would be the profit of all of that? Clearly: “one vanished and one appeared”, so everything will remain the same counting in pieces – 30 were, 30 are there. But “in length” we will win clearly exactly 1 meter, and we need to win 10 meters so we must exchange exactly 10 such 3 meter sticks, sorry, pieces, and then everything would be done.

So we conclude – at this time, formally speaking, without writing any equation but in fact doing the same in mind, that we have 20 pieces of 3 meters lengths and 10 of 4 meters so in order to prepare from all that the material wanted or “unified only 1 meter long pieces” again we will need

$$20 \cdot 2 + 10 \cdot 3 = 70$$

cuts.

This entire assortment was prepared for sweet Molly Melone under motto – simple girls – understandable deeds.

But what we’ve seen was truly much more than our everyday fantasy could ever imagine. Of course we understood after some hesitation what’s going on but in the same time we must frankly add and confess that our respect for Molly increased considerably.

Instead of cutting Molly went ahead restoring this 100-meter log putting all pieces as if they were together or originally – all in one line. Of course, after she completed it we could state that we still see all these 29 “touching” places of these pieces. But we have nothing against the imagination that this is again entire 100 meters long piece.

And now Molly told to all passengers present on that Dublin street that now she needs 99 cuts in order to have

100 pieces, all of them being 1 meter long, from that original log.

She told she understands that 29 cuts are already made, so in fact only

$$99 - 29 = 70$$

cuts will be needed.

This girl did it even without knowing how many 3 meters and 4 meter pieces were lying on the scene.

That's that!

The answer.

70 cuts will be necessary and enough.

A. D. 2003, Grades 5 and even 6, problem 5. Serious-remaining TOM and JERRY deciding about their exciting problem, which one from these 9 numbers will remain if each of them already dismissed some 4 of these numbers and the sum of numbers removed by TOM is thrice as big as the sum of numbers removed by JERRY – or vice versa – it makes no difference for the number, which will remain!

Instead of solution or thoughts we report about some ideas of our already skilled friends

Each experienced adviser, like we already are, would inform these actors if they will ask about it that the thing which could be stated at once is that the sum of all numbers, which were removed by both of them is clearly and undoubtedly divisible by 4.

That's easy to feel and even to check:

If JERRY removed "the sum"

$$N$$

then TOM removed the sum, which is

$$3N,$$

so *the sum of numbers removed by both of them is*

$$N + 3N = 4N$$

And what could be deduced from that obvious (but useful, you'll just see!) statement?

Nobody could answer that question better than Tom and Jerry but actually now they are busy counting once again the sum ***of all*** of these presented numbers

$$1 + 3 + 4 + 6 + 8 + 9 + 11 + 12 + 16 = 70$$

After that, what could be stated? The number 70 is neither so famous nor so large that everything could be stated at once and known for advance.

And now the simplest possible insight is deciding: let's write it down.

If the remaining number is 1, then the sum of remaining numbers (or exactly these, which, according to the parabola, were removed by TOM or by JERRY), would be 70 – 1 = 69. So let us present all these exciting possibilities:

If number 1 overcomes then the sum of displaced numbers is 70 – 1 = 69;

If number 3 overcomes then the sum of displaced numbers is 70 – 3 = 67;

If number 4 overcomes then the sum of displaced numbers is 70 – 4 = 66;

If number 6 overcomes then the sum of displaced numbers is 70 – 6 = 64;

If number 8 overcomes then the sum of displaced numbers is 70 – 8 = 62;

If number 9 overcomes then the sum of displaced numbers is 70 – 9 = 61;

If number 11 overcomes then the sum of displaced numbers is 70 – 11 = 59;

If number 12 overcomes then the sum of displaced numbers is 70 – 12 = 58;

If number 16 overcomes then the sum of displaced numbers is 70 – 16 = 54;

But from all of these sums

54, 58, 59, 61, 62, 64, 66, 67 and 69

only 64 is divisible by 4 and all others aren't, so exactly the number 6 could remain and we believe that this will undoubtedly immediately established after all these correct historical investigations.

The answer.

"6" is the number, which will remain. "6" will overcome – no other number.

A.D. 2003, Grades 7 and even 8, problem 1. Granny's desire is to list all 5-digital and 10-digital numbers with only 2's and 5's in their decimal expression and with the strict condition that no two 2's can ever stand beside.

Solution with some reflections

In Advisory Board we concluded at once that with such 5-digital numbers no considerations for Granny will be necessary – he will simply list them all and that will do.

And exactly so had happened. Granny understands that the first digit of any *two-five-two number* is either 2 or 5 – both are possible.

So we have either

2*xxxx*

or

5xxxx.

In any two-five-two number there are no two neighboring 2's so there are 3 (but not 4 as it is in the case without restrictions) possibilities taking also the second digit into account:

25*xxx,* 52*xxx and* 55*xxx.*

Writing consecutively and systematically, in the third step we get the possibilities.

252*xx*, 255*xx*, 525*xx*, 552*xx* and 555*xx*.

Continuing, on the last but end step we'll get

2525*x*, 2552*x*, 2555*x*, 5252*x*, 5255*x*, 5525*x*, 5552*x* and 5555*x*.

And finally the complete two-five two list containing all such 5-digital numbers is

25252, 25255, 25525, 25552, 25555, 52525, 52552, 52555, 55252, 55255, 55525, 55552 and 55555.

So there are exactly 13 of them.

Now concerning such a numbers containing 10 digits it could be stated that it is still possible to write them down "by hand".

But it is not so interesting. And, writing down similar numbers consisting of 5 digits, we've already learned a lot.

So let us speak "theoretically" now: assume that there are

S(10)

suitable numbers. Needless to say is that each of them is ending either in 2 or in 5.

Similarly on the last but one step there were

S(9)

and two steps before the end there were

S(8)

suitable numbers.

Now it would be very nice to notice some connections between these numbers, possibly of the simplest kind and nature.

The sample of suitable 10-digital integers, the number of which is S(10), consists of 2 natural sub-samples.

The first is the collection of all 10-digital integers, which end in 2, and the second is the collection of those, which end in 5.

The number of suitable numbers ending in 5 is equal to the number S(9) because 5 can be written at the end of any suitable 9-digital number getting suitable 10-digital number.

A bit more difficult is to understand that the number of suitable 10-digital integers, which end in 2, is exactly S(8).

Then we will get that

$$S(10) = S(9) + S(8).$$

But if we replace 10 by another number, say, 6, we will have similarly that

$$S(6) = S(5) + S(4).$$

It is even possible to replace 10 by n,
getting in the same way

$$S(n) = S(n-1) + S(n-2), \text{ where, of course, } n \geq 3$$

But in the first part we've had that

$$S(5) = 13$$

and similarly

$$S(4) = 8.$$

Also it was established that S(3) = 5 and S(2) = 3.

Applying consequently we state that from

$$S(6) = S(5) + S(4)$$

it follows that

$$S(6) = 13 + 8 = 21.$$

Now turn by turn

$$S(7) = S(6) + S(5) = 21 + 13 = 34,$$
$$S(8) = S(7) + S(6) = 34 + 21 = 55,$$
$$S(9) = S(8) + S(7) = 55 + 34 = 89,$$
$$S(10) = S(9) + S(8) = 89 + 55 = 144.$$

The answer.

Granny will get 13 suitable 5-digital numbers and 144 suitable 10-digital integers.

A.D. 2003, Grades 7 and even 8, problem 2. Solution with more than 1 drawing and simple consideration and reflections

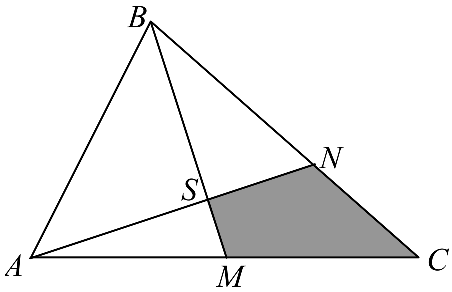
The recent intellectual deeds of the famous broker Ali Baba included his famous split in the triangle-40. Here is that problem.

On the median *BM* of the triangle *ABC* such a point *S* is chosen that *BS* = 3*SM*. A line passing through the points *A* and *S* intersects the side *BC* at a point *N*.

Find the area of the quadrangle *CMSN* if the area of the initial triangle *ABC* is 40.

Let us make the Figure 1 with the picture of initial state.

Figure 1.



Now let us prepare – do not forget that we are from the Advisory Board of Ali Baba – another picture with the larger number of parcels. For that we mark the points *P* and *R* on former *BS*, which is thrice as big as *SM*, dividing now it in three equal parts

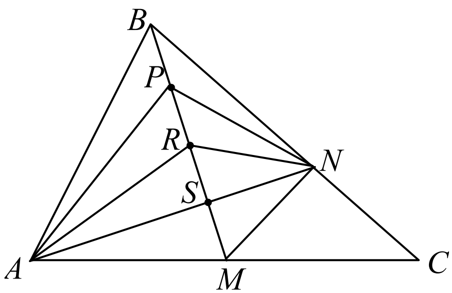
$$BP = PR = RS.$$

So the median *BM* is now already divided into 4 equal parts – the fourth of which is former *SM*:

$$BP = PR = RS = SM.$$

Now we join the point *N* with these points *M, S, R and P* getting many parcels, and all of them are indicated in the Figure 2.

Figure 2.



Now we are going to mark them using as few different letters as possible – or we intend to mark the triangle parcels of equal area with the same letters and even with particular numbers - if only we could succeed in understanding. Two main phrases will now be pronounced and permanently repeated.

The first of them will be that the 2 triangles are of equal area if we can show the their bases are of equal length (and belong to the same common line) and their third vertex is the same point (then they will share a common altitude from that point!)

Now in Figure 2 the triangle *ABM* is naturally divided into 4 triangles *AMS*, *ASR*, *ARP* and *APB*. All 4 of these triangles have the base of the same length laying on the median *BM*:

$$BP = PR = RS = SM$$

and share the same third vertex at *A*, so they have the same altitude from that point and so they all will have the same area, which we will denote by ♣.

Reasoning identically we will have on another side of *BM* the triangle *NMB* divided into 4 triangles

NMS, NSR, NRP and *NPB*

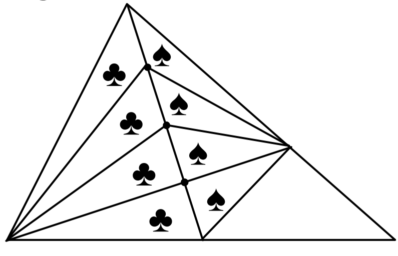
with the same equal bases

$$BP = PR = RS = SM$$

The common vertex of them all is at *N*, and their equal areas will be denoted by ♠ each.

Let us indicate it in Figure 3.

Figure 3.



Now we can state that “almost all” area of triangle *ABC* is covered by these ♣ and ♠. The only uncovered area is that of triangle *CMN*. We could ask now the following question.

Are there in the triangle *ABC* some smaller (sub)-triangles with the same area as *CMN*?

After some consideration we see that the answer is “yes” – such is the triangle *AMN*. This is indeed so because *BM* being median from *B* means that

$$AM = MC$$

on the side *AC* and the third common vertex naturally in *N*.

And what is the area of that *AMN*? From all our markings, from all these ♠ and ♣ it is clear that the area of *AMN* is

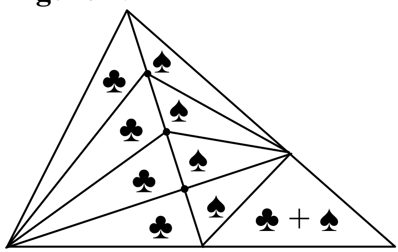
♠ + ♣.

So now the area of that only non-marked triangle *MCN* is also

♠ + ♣.

In Figure 4 will indicate now the complete partition of our triangle using the only measures ♠ and ♣.

Figure 4.



Now it is necessary only to complete the solution.

The useful link is to state that on each side of median *BM* there is as much of area of initial triangle *ABC* as on the other. It was mentioned that that area of *ABC* is 40.

So there is area 20 on each side; so 20 is the area of the triangle *ABM*, and 20 is also the area of the triangle *CBM*. But on one side there are 4 small triangles each of which is marked with the same ♣. In other words, the area of the triangle ABM is 4♣ or 20.

But

$$4♣ = 20$$

means

$$♣ = 5.$$

On the other side of median BM there 5 triangles with areas

$$♠, ♠, ♠, ♠, ♠ + ♣ .$$

That makes the total area being

$$5♠ + ♣$$

or altogether also 20:

$$5♠ + ♣ = 20$$

But

$$♣ = 5$$

leaves

$$5♠ = 20 – 5 = 15$$

giving finally

$$♠ = 3.$$

Now in the triangle *ABC* we can compute almost everything we would like to. We need the area of the *CMSN*. This is exactly

$$♠ + (♠ + ♣)$$

or

$$3 + (3 + 5)$$

giving 11 as an answer.

The answer.

The required area of CMSN is 11.

A. D. 2003, Grades 7 and 8, problem 3. Solution and related considerations

The Lady is of course essentially right – have you ever met a large number on the country road?

Still if we could imagine that we are able in our monitor to enjoy the overview of every country road that ever existed then our hopes would be considerably more realistic.

So at any rate our advisory board started generating ideas and we are happy to announce that according to our knowledge the grandson of Mr. Sherlock Holmes dealing under conspiracy name Oneth Wothree overtook the responsibility for that whole affair. We only add that Oneth wishes one day or another to enter the famous UCL in order to have a lecture course by the famous professor Jayne.

He started in an astonishingly simple way – he took the block

2003

and started to build be bigger numbers – firstly

20032003….

then

200320032003….

So far everything is understandable – putting together these "2003-blocks" Oneth will never lose the divisibility by 2003 and will increase the digit sum – with each block by 5 units making it correspondingly

5, 10, 15, 20,... .

But the dream-number 2003 will never appear as a sum here so on one day or another Don Quichote with Oneth would be forced to invent something else.

Indeed after putting of

400

such 2003-blocks together the number

200320032003…2003

would appear on some country road and that number would count

$$400 \cdot 4 = 1600$$

digits and its sum of digits will be already near to what is needed:

$$400 \cdot 5 = 2000$$

but with the next similar step this sum of digits will jump over that desired 2003 and further also will be never meet it again.

But actually at that place Oneth also stopped. His took out the first 2003-block from that huge number and started to consider its multiples

2003, 4006, 6009, 8012, 10015, 12018, 14021, 16024, 18027, 20030

writing their digit sums underneath:

5, 10, 15, 11, 7, 12, 8, 13, 18, 5….

Then he took the new block 14021 (with the sum of digits 8, mentioned we in the monitor of the advisory board), and put it back instead of the former block 2003. Aha, so now he'll get the modified number

14021 2003 2003….. 2003

consisting of again 400 such almost-all-2003-blocks with the sum of digits 2003 instead of 2000 as it was. And he won't lose the divisibility by 2003 because he was regarding the multiples of 2003.

Answer as it was written by the hand of Don Quixote:

There are plenty such a number on the world – on my own way I met once the number

14021 2003 2003 ….. 2003 .

399 blocks "2003"

A. D. 2003, Grades 7 and 8, problem 4.
The natural wish of John the Optimist is to get as many as possible integer numbers from 11 fractions in which each number from 1 to 22 is used either as numerator or as denominator.
Solution and related details

The dual problem for Jack the pessimist would be to get as few from 11 fractions to be integers – again employing all numbers 1 to 22 exactly once – would be easy. That easy answer would be – the case with no integer numbers from all of those 11 fractions is possible.

It would be enough then to refer to such a sample of 11 fractions:

$$\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \frac{9}{10}, \frac{11}{12}, \frac{13}{14}, \frac{15}{16}, \frac{17}{18}, \frac{19}{20}, \frac{21}{22}$$

with clearly none of them being an integer number.

It would be possible and natural to imagine that the corresponding possible "dual" alternative for John the Optimist would be to hope that it is possible to make all 11 of these fractions to be integers.

Unfortunately that is not possible.

Why?

Because among all these numbers there are some big prime numbers – 13, 17 and 19.

What do they do?

Their crime is that from the fractions in which they are participating at least one is always not an integer number.

What are the reasons for that?

The reasons for that might be explained as follows.

If any of these 3 numbers 13, 17, 19 is a denominator then such fraction can't be an integer.

If some 2 of these 3 numbers 13, 17, 19 are numerators then at least one of corresponding fractions is not an integer.

Really, if some 2 of these 3 numbers 13, 17 and 19 are numerators then the only possibility to get an integer from such a fraction is to have 1 in denominator – so that at least one of remaining fractions wouldn't be an integer.

It means that not all fractions can be integers

Consequently at most 10 of them can be integers.

After a while John the Optimist presented 11 fractions, and 10 among them were indeed integers – all with exception of the last fraction:

$$\frac{13}{1}, \frac{10}{2}, \frac{21}{3}, \frac{20}{4}, \frac{15}{5}, \frac{12}{6}, \frac{14}{7}, \frac{16}{8}, \frac{18}{9}, \frac{22}{11}, \frac{19}{17}.$$

The answer.

John the Optimist cannot make all of these 11 fractions to be integers. He can make all of them but one integer numbers:

$$\mathbf{\frac{13}{1}, \frac{10}{2}, \frac{21}{3}, \frac{20}{4}, \frac{15}{5}, \frac{12}{6}, \frac{14}{7}, \frac{16}{8}, \frac{18}{9}, \frac{22}{11}, \frac{19}{17}}$$

(only that last fraction is not an integer!).