The LAIMA series

Romualdas KAŠUBA

WHAT TO DO WHEN YOU DON’T KNOW WHAT TO DO?

PART II

R. Kašuba. What to do when you don’t know what to do?

Part II.

Rīga: Mācību grāmata, 2007. – 166 pp.

The book analyses psychological aspects of problem solving on the basis of contest problems for junior (4^{th} – 9^{th} Grades) students. Nevertheless, the approaches discussed are of value also for highest grades, for teachers, problem composers etc. The text can be used by all those who are preparing to research in mathematics and/ or to math contests.

The final version was prepared by Ms. Dace Bonka.

ISBN 9984-18-284-3

Reģ. apl. No. 50003107501

Iespiests SIA „Mācību grāmata”, Raiņa bulv. 19, Rīgā, LV-1586, tel./fax. 7325322

To Ieva - a daughter and a friend.

The author.

FOR THE ENGLISH EDITION

This is the second part of the book [1], which under the same name appeared in Lithuanian in A. D. 2006. The English translation of it, made by the author, was splitted in two parts in a rather mechanical way. First part of it appeared in 2006 [2]. It might be mentioned that these two parts can be regarded either as two parts of the some imaginary unit gathered under the given title or also as well as two independent texts at least in the sense that both texts could be read in any order.

The only thing which now keeps these two parts together beside the nature of regarded problems is the double numeration of chapters.

The author would like again to express his gratitude to the organizers and publishers of LAIMA series for publishing Parts I and II in English. It is great joy and rather pleasant responsibility.

Frankly speaking there are some persons to whom I would like to express my thanks much more than once.

First person to whom I would like to address my multithanks is Professor Benedikt JOHANNESSON who

according to my understanding was primus inter pares in inventing that whole happy affair, which name is LAIMA series. Another thing worth mentioning is his way of being and manner of speaking, which I sometimes was trying to imitate.

I am especially thankful to Professor Agnis ANDŽĀNS also for constant help and delicate support. Professor Andžāns was also the first official reviewer of the Lithuanian edition. It could be only repeated that without his inspiration and practical care this translation would probably never appear.

I am also thankful to Professor of patience Dace BONKA who was working with the Part I and hopefully - also with the Part II.

Romualdas Kašuba

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ABOUT LAIMA SERIES

In 1990 international team competition “Baltic Way” was organized for the first time. The competition gained its name from the mass action in August, 1989, when over a million of people stood hand by hand along the road Tallin - Riga - Vilnius, demonstrating their will for freedom.

Today “Baltic Way” has all the countries around the Baltic Sea (and also Iceland) as its participants. Inviting Iceland is a special case remembering that it was the first country all over the world, which officially recognized the independence of Lithuania, Latvia and Estonia in 1991.

The “Baltic Way” competition has given rise also to other mathematical activities. One of them is project LAIMA (Latvian - Icelandic Mathematics project). Its aim is to publish a series of books covering all essential topics in the area of mathematical competitions.

Mathematical olympiads today have become an important and essential part of education system. In some sense they provide high standards for teaching mathematics on advanced level. Many outstanding scientists are involved in problem composing for competitions. Therefore “olympiad curricula”, considered all over the world, is a good reflection of important mathematical ideas at elementary level.

At our opinion there are relatively few basic ideas and relatively few important topics which cover almost all what international mathematical community has recognized as worth to be included regularly in the search and promoting of young talents. This (clearly subjective) opinion is reflected in the list of teaching aids which are to be prepared within LAIMA project.

Twenty books have been published so far in Latvian. They are also available electronically at the web - page of Latvian Education Informatization System (LIIS) http://www.liis.lv. As LAIMA is rather a process than a project there is no idea of final date; many of already published teaching aids are second and third versions and will be extended regularly.

Benedikt Johannesson, the President of Icelandic Society of mathematics, inspired LAIMA project in 1996. Being the co-author of many LAIMA publications, he was also the main sponsor of the project for many years.

This book is the fifth LAIMA publication in English. It was sponsored by the Scandinavian foundation “Nord Plus Neighbours”.

BREAKTHROUGH I (XXV).
THE GLAMOUR OF SIMPLE THINGS OR ADVENTURES BY DIVIDING CABBAGES

"Two added to one – if that could be done,"
It said, "with one's fingers and thumbs!"
Recollecting with tears how, in earlier years,
It had taken no pains with its sums.

Sometimes the problems arise exactly so as it was expressed in some good temperate advertising where the drivers were asked to be especially careful with children on the street because "they appear from nowhere". Here we propose one exercise with some elements of that sort and kind.

To the brother hare another its twin brother arrived exactly in the moment when on the table three cabbage-heads were laying and they wages were known as well being 300, 500 and 700 grams exactly. The twin brother as the real guest had a forehand in choosing and so starting the whole process of consuming. After he'd already chosen the cabbage head the host also took one and without losing any second also started the process of devouring. They both are consuming the cabbage mass with the same constant speed. With which cabbage-head should the guest-brother start in order to be able to consume more or at least the half of the whole cabbage mass?

The common sense indicates clearly that the twin hare-brother as a guest is in favour possessing the privilege of selecting first. So for him getting less than the half of the whole cabbage mass wouldn't be especially skilful. Simply speaking it would be very bad. Because starting first means that you have more possibilities to choose and are expected to be more successful, and you are clearly not expected to get less then a half!

And that is very important exactly in all cases when you are not so sure what to do.

That is connected also with the title of our book.

Our book tries to give you some advices in order how you can achieve some real progress by undertaking simple efforts and applying uncomplicated common sense methods.

You can do more when you would ever expect simply by starting and doing something and not just going on complaining all the time long how difficult it is to achieve something in our wicked world.

Don't cry for me, Argentina.

Let's at this place leave for a while our precious considerations and try to regard all the possible choices of the guest hare twin brother.

His first desire could be a clear desire to start selecting of course the biggest cabbage head or that 700-gram one. Then in the very same moment another twin brother will behave as the most modest host in the whole world responding with the choice of the smallest 300-gram head. He will consume it completely and then will quietly continue – again losing no time – with the only remaining 500-gram head left. The host could of course change these 300 and 500-gram heads. In both cases the host would consume more than the half of the whole cabbage mass because clearly

$$300 + 500 = 800 > 700.$$

This indicates that our cordial advice for the guest hare twin brother would be to remember the importance of being modest or repeating “Don’t touch the biggest head without the clear need”. Following that advice that guest could take the 300-gram head.

Then in the case when the host answers with the choice of 500-gram head the guest then will also consume the remaining 700-gram head making his total consumed weight to be exactly 1000 g.

If the host would answer with 700-gram piece then the guest brother will get also the remaining 500-gram piece achieving total consumption of 300 + 500 = 800 grams. This is not as much as in the first case but still more than the half of total cabbage mass.

Again completely bad for the guest would be the remaining third start possibility or selecting the 500-gram piece because then the host brother after most modest answer or after taking the smallest 300-piece would get also the remaining biggest 700 cabbage-head or consume 300 + 700 = 1000g of cabbages.

The conclusion is that having three heads of cabbages with weights 300, 500 and 700 grams the guest brother ought to start in most modest way simply taking the smallest piece.

It’s a pity but we are not able to draw immediately the following conclusion in the form of maxim

Start always with the easiest head when you’ll consuming cabbages.

This is not true even if we remain in the 3-head system.

The reader would easily find some examples illustrating that. If you are not able to spend enough time for constructing an example yourself, then we would highly recommend you to regard the 3-head system of weights 300, 500 and 1700 respectively.

After all that experience our maxim could be

Start always with the easiest head unless you are convinced that there are better choices of getting more cabbages.

But now let us go back to the realities and see whether during the second visit when the guest and host will change roles the first eater could get at least a half of proposed cabbages starting modestly. We already noticed that this depends on the concrete weights. We will always try to go on giving precious advises.

During that honourable event the former guest now appears in the role of host. On the table he sees now already 4 cabbages or 300-, 500-, 700- and 900- gram heads. That increasing of the numbers of cabbage heads demonstrates also that the quality of the hare's life is also improving.

As it was already mentioned now the former host is the guest and as such has forehand to choose first. The main practical question is whether in this case also the modest start will again bring for the guest more than the half of the whole consumed cabbage mass? We understand perfectly well that all these guest-host roles are the same as indicating who's takes first.

There are 4 cabbage-heads waiting for their preferences.

First possible choice or take the biggest head first.

If the guest would start doing so and start consuming 900-piece then he would prove himself neither especially modest nor skilful. His hopes doing so to achieve at least the half of the total cabbage mass will be never fulfilled.

What will happen?

The host will act modestly and effectively taking firstly smallest 300-piece, then again modestly selecting the smallest remaining 500-piece and finally - what to do when there is no other head left – he is simply forced to take the remaining huge 700-piece.

So selecting the heaviest head is again not the best solution.

Second possible choice or try to take the second biggest head.

As it was already noticed the greedy choice for the guest proved itself to be a bad choice. If the guest starts more modestly by taking just second heaviest piece weighting 700 grams that choice would prove itself also to be not especially useful because then the host's response will be 500-gram head getting afterwards also the biggest 900-head and so consuming (much) more than the half of the whole cabbage mass because

$$500 + 900 > 300 + 700.$$

The third choice would be to start with the smallest piece.

Well, and what happens then if the guest starts in the most modest way selecting the smallest 300-head? It looks so nice and demonstrates good manners. Will it bring more than first two choices? The possible advice for the host to drop away all shyness and catch the 900 piece immediately would be bad preference because then the guest in the second turn takes quietly 500 gram piece and will surely get also the remaining 700 gram piece in such a way getting (much) more than the half of mass.

Other choices for the host or taking any of 500 or 700 pieces give to both of them a half of cabbages.

So the third or the modest choice of guest is better than the other two.

Finally what about the last or fourth choice starting with the third biggest piece?

Then the best answer of the host is to take the 300-piece and only after that the biggest 900-gram piece. Then they both will get the half of mass.

So in that case selecting properly brings them both to the "fifty-fifty situation".

Some conclusions after first two visits: not cabbages themselves but numbers expressing their weights rule the process of successful consuming.

Roughly speaking we landed by the conclusion that when two hare brothers were consuming cabbage heads we've found some relations between their weights ruling the more or less successful results of consuming.

Even the smallest proper step towards the aim increases our hope to finish the task thrice.

Recall that there where two consuming rounds of two hare twin brothers:

First round was their consuming of 3 heads weighting 300, 500 and 700 grams.

Second one was their consuming of 4 pieces weighting 300, 500, 700 and 900 grams.

What a philosophy follows from all we've seen?

Firstly perhaps that we could omit two noughts present in all our weights and discuss weights or numbers 3, 5 and 7 in the first and numbers 3, 5, 7 and 9 in the second case. The intrigue would remain just as it had been but the numbers characterizing it would be (hundred times) less. That is convenient from the psychological point of view because we are much more self-confident when dealing with small numbers. We are not at least a bit afraid of them. Still we understand, of course, that even small numbers can sometimes prepare to us some surprises as well as sometimes even our best friends do.

In the first case, if we are to choose first, then the inequality

$$3 + 5 > 7$$

indicates that, taking the smallest head, we will (independently from the choice of the partner) get also second number which guarantees us more than a half of the whole consumed mass. Recall that all that is due to numerical inequality $3 + 5 > 7$ *and to the rules of consumption of course.*

In the second case dealing with four numbers 3, 5, 7 and 9 during all that process we never let out of our minds the inequalities

$$3 + 5 < 9, \quad 5 + 7 > 9$$

and clearly see that

$$3 + 9 = 5 + 7.$$

This also explains why 9 as the first turn is not good but just worst decision –because the partner gets all remaining 3 numbers making total

$$3 + 5 + 7 = 15$$

– this is much more than the half of all – so greedy algorithm didn't work again!

We understand at once also why 7 as the first turn is also not the best possible choice because it will be followed by selection of 5 with following getting of 9, and

$$5 + 9 > 7 + 3.$$

This second choice is slightly better because it leaves for the second hare 14 units of weight and not just 15.

These both situations allow slight generalization by stating that all considerations word by word would remain the same replacing

(ii) *numbers 3, 5 and 7 by any numbers A, B and C with $A < B < C$ and $A + B > C$.*

Again selecting the smallest number A in the first turn I will in any way get another of remaining number in my second turn also independently after any choice of my partner.

You see that we forgot the hairs – only the two players – you and me – are now to be seen on stage.

(ii) *numbers 3, 5, 7 and 9 with any numbers A, B, C and D with $A < B < C < D$*

with

$$A + B > C,$$
$$B + C > D,$$

and

$$A + D = B + C.$$

All considerations can now be repeated word by word.

Return to the following new situations in which our twin brothers are involved, look how are they consuming, and generalize.

We propose for the further considerations some other collections of four cabbage-heads. It is very simple to describe their weights and probably not so simple to solve.

In the former second set of weights 300, 500, 700 and 900 we change the heaviest head of 900 gram by the easier one weighing 800 grams.

Yet we propose also the 4-head system of 200, 210, 300 and 310 grams.

That formulation of the problem with all the numbers is due to Lomonossow multi-discipline contest where many nice and original but still highly excessive problems for all who are eager to deepen the power of theirs minds are presented.

BREAKTHROUGH II (XXVI).
CHILDREN TALKS OR FEATURES OF THE FUTURE

The children aren't happy with nothing to ignore
And that is what (or why?) the parents are created for.

Once upon a time in the strict place there lived - all in the same family - 4 children whose names were Aurora, Justus, Andrew and Johnny. We all are inclined and eager to believe that when they will grow-up, they will prove themselves to be good-hearted, funny and patient just as they are.

Who knows, maybe an adventure similar to that which we are going to tell has also taken place with other children involved.

On a sunny Sunday this quadruple was sitting in the near of their Grandma waiting for the Sunday meal and for the fairy tale. When you are waiting then your sack of sweets is of great value. While the Grandma who was very serious when dealing with adult persons was turning about in kitchen preparing her big potato pancake, then… exactly then it had happen. It was a huge breakthrough of fraternal kindness they all were overtaken by, a kindness, which so often assists also serious children.

Suddenly the elder sister Aurora softly and suddenly told to her resting brothers:

- You, my brethrens, show me now how much sweets each of you hold in your sack, and I will make that each of you will find that he has already twice as much. I repeat you – each of you will have twice as much. Do you agree? Can you believe?

There was no reason to refuse and brothers agreed.

And so it happened. It was a miracle. It was a miracle with real content.

Aurora gave to every brother from her sack of sweets exactly as much sweets as each of them proved to possess.

It's wondering that the miracles happen. But can you believe that miracles do happen in series – one after another?

Can you believe it or not – but it happened!

Inspired by their sister's nobleness the brothers did the same doubling of sweets. They did it one after another.

Johnny was the first who dared to repeat was Aurora had just done. This was again a miracle. This was again a miracle with real content. Johnny doubled the sweets from his own supplies.

Then Andrew did the same. For the youngest Justus there was practically no other choice left. He couldn't look worse. He did the same. Doubled the real content again. That of all other brothers and the sister, of course.

After everything was over they regarded each other with notable surprise and somehow compared their number of sweets they actually had after these 4 doublings. To their great surprise it turned out that they all were equal. It turned out that every one had exactly the same quantity of sweets – 16 sweets each.

It is humanly interesting to ask whether knowing the final state we could determine how many sweets were there in the sack of each child before these doublings started.

What was at the beginning?

Who gained and who lost?

And almost unavoidable question:

Who of them proved to be the noblest one?

And (the) last step of our deeds will be the first step our solving.

This is one of most natural or fruitful ideas. That is similar to reversing the film. That is what you could do if you really want to make us understand how the things were developing looking back from the final picture. If we are able to pursue how the things were developing then it is often possible to describe how did they start.

To do this just in one move usually is not possible. But we can do it in several moves or step by step – life teaches us to behave modestly if we are eager to achieve some progress.

To reach the initial state from the final position is not always possible, but it's always worth trying. It could be asked why it's so precious and worth trying? The answer could be because

The truth is powerful, attractive and convincing.

We will try to find the truth also in this described case of the sweet distribution.

So recall now the final state of things: *after the last doubling of sweets which was providing by the youngest* *brother Justus each of them proved to possess exactly 16 sweets.*

This could be described as an idyllic state of things where all of them are equal with regard to the number of sweets. Of course it is not so clear – you never know - if all of them are equally content regarding that circumstance and remembering how different the numbers of sweet probably were at the very beginning. Now the quantitative final picture of that what they have is fully described in following table:

AURORA	JOHNNY	ANDREW	JUSTUS
16	16	16	16

Now we'll try to do the first step backwards reversing the film, which supposedly will tell us everything about the whole development or all about these 4 doublings. Our global aim is to establish the initial number of sweet of each of these four children just before they started distributing sweets to all remaining children. They did it one by one.

First was Aurora.

Second was Johnny.

Third was Andrew.

Justus undertook final doubling.

The last step was the doubling due to Justus. Is it possible to display the things back? Can we establish how many sweets they have had before Justus had undertaken his doubling with the real content? It seems that this is possible.

We already mentioned that the total number of sweets remains the same during this whole noble adventure. Sweets only change hands.

During the whole process they all have

$$16 + 16 + 16 + 16 = 64$$

sweets. We believe that all of them were involved to that distribution procedure so deeply and thoroughly that they didn't eat anything including the sweets at all. This has also something in common with the so-called real content of similar problems. So let's imagine and assume that there was no consuming during the whole adventure.

So Justus gave to every one as much sweets as any of the resting 3 has had. And after the last doubling each of the four has 16 sweets, so Justus must have been given of each of them exactly the half of it or $16 : 2 = 8$ sweets to each of the remaining three.

So the whole number of sweets given away by Justus was

$$8 \cdot 3 = 24.$$

Because after all he also has 16 sweets so before his doubling he had

$$16 + 24 = 40$$

sweets and was the richest or most "sweetest" one.

It means that one step before the final state the distribution of sweets was as described in the table:

AURORA	JOHNNY	ANDREW	JUSTUS
8	8	8	40

Now we would like to develop our success and to deepen our insights and to try to establish what was the distribution of sweets two steps before the final state. Two steps before the final distribution is the "middle" state because it's exactly a state after two first doublings.

The last but one doubling was Andrew's honour and trouble. He also, as it became already usual, gave to resting three exactly the half of sweets that they actually had. That means he must have given $40 : 2 = 20$ sweets to Justus, and 8 : 2 = 4 to both Aurora and Johnny. So Andrew had given away

$$20 + 4 + 4 = 28$$

sweets so just before doing that he must have had

$$8 + (20 + 4 + 4) = 8 + 28 = 36$$

sweets. So we did the second step returning back to the initial state, and the distribution of sweets is as it is shown in the table below:

AURORA	JOHNNY	ANDREW	JUSTUS
4	4	36	20

We did two steps backwards. We are exactly in the middle of our noble process. We are expected to do the remaining 2 steps as well. We will combine them both in one table, 2 in 1.

In the second row of the table below their names we indicate the distribution of sweets just before the Johnny’s doubling.

In the third row we will indicate what the situation was just before Aurora’s doubling. She generated that noble idea. And the situation before Aurora’s doubling is the initial state which we were so eager to display.

AURORA	JOHNNY	ANDREW	JUSTUS
4:2=2	64-(2+18+10)=34	36:2=18	20:2=10
64-(17+9+5)=33	34:2=17	18:2=9	10:2=5

So what have we achieved? We succeeded in displaying the whole picture moving backwards step by step from the final situation to the initial one. It was a reverse joy in 4 parts.

Some avoidable philosophical reflection ought to also be mentioned.

First of all, their sister Aurora proved also to be the noblest between then. We still remember that it was she who gave the idea.

She was the noblest one because at very beginning she had more than the half of the total number of sweets or 33 of them.

Comparing this with final situation which could be completely characterized by the words “all are equal now”, we conclude that Aurora has given away $33 - 16 = 17$ sweets or slightly more than she finally possesses.

Now concerning Johnny. Johnny’s number remained practically the same as it had been. It was 17 and now it is 16. Strictly speaking he is also the donator. He has now less then he had at the beginning.

Well, it remains to notice that Andrew had earned $16 – 9 = 7$ and Justus even more or $16 – 5 = 11$ sweets.

Perhaps it is the best because they are the youngest ones.

BREAKTHROUGH III (XXVII). AGAIN MONEY, STAKE AND RACES

Each of us has heard a lot or more than enough about the races, risk and hazard and horses arriving at finish first and even about the scandals assisting such entertainments.

Arrived the neighbour Peter, the known storyteller, and announced that in Math village the racecourse will be inaugurated. Firstly there will be only three racing horses. At the first glimpse his words looked not so convincing. But further some strange things occurred.

You ought to know him better. Peter knew already even the names of these three horses. According to him their names were Trumpet, Drum and Panpipe. This sounded more like music instruments rather than horses. The Trumpet was supposed to be the best among these three; two others Drum and Panpipe were not so brave.

Because Trumpet was claimed to be the best horse it’s no wonder the bets upon him by the racing bookmakers actually stand by 1 : 1. Peter eagerly explained that this means if your bet upon him was 100 Euro and Trumpet arrives first then you firstly get your 100 back and you’ll be given another 100 just to honour your deep insight.

But if Trumpet won’t be first and you’ve laid 100 upon him then you can forget your money.

The chances of Drum actually were 1 : 4, and these of Panpipe even by 1 : 5. This naturally means, that if you risked to support your idea that Drum will arrive first at finish by 100 Euro and Drum indeed crosses first the finish line then you will not only get your 100 back but will be also rewarded by four hundreds. In the similar thing will happen with Panpipe you would get even five not just four additional hundreds. This is not astonishing because he is assumed to be the weakest horse and perhaps seldom if ever crosses the finish line first.

Needless to repeat that if the horse you’ve chosen doesn’t come first then better again forget the money you laid on.

What could be our attitude to Peter’s words? Let us think a bit. We have the strange idea that by that racecourse we can really earn some money. Could it be so? Are they freshman on the area?

Let us try to make a bet and draw conclusions.

Assume that our bet on Trumpet is A monetary units, our bet on Drum – B monetary units, and let our bet upon Panpipe be correspondingly C monetary units.

Then it is clear that in the case when Trumpet comes first – recall that only these 3 horses are competing – then we'll earn $2A$ monetary units (they will give us our A units we've laid upon back and give us another A units for our successful risk). In the case when Drum arrives first at finish line we will earn $5B$ monetary units (the B units we've laid and additionally got fourfold of the laid sum or $4B$ for our risk). Finally if Panpipe will first cross the finish line – what is least believable – according to the estimation of his chances - then we will get our C units back together with other 5C monetary units as an honourable payment for our successful risk.

So we laid

$$A + B + C$$

monetary units, say, in Euro or even in dollars, and if we expect to make the positive balance independently which horse will cross first the finish line, then in any case the sum we get must exceed our money amount which we left in the racehorse booking office. So it ought to be

$$\begin{cases} 2A > A + B + C \\ 5B > A + B + C \\ 6C > A + B + C \end{cases},$$

or

$$\begin{cases} A > B + C \\ 4B > A + C \\ 5C > A + B \end{cases}$$

It is not so complicated to find some solutions for that system of inequalities, e.g.,

$$A = 10,\ \ B = C = 4,$$

or if we choose to operate with “round” numbers then we can take 50, 20 and 20 as well.

That is, if we laid 50 monetary units on Trumpet, 20 upon Drum and also 20 units for Panpipe then we left

$$50 + 20 + 20 = 90$$

monetary units in booking office.

Now in the case if Trumpet wins we would get our 50 back and also another 50 as well, or spending 90, we get 100.

In a case when Drum wins our win would be also $5 \cdot 20 = 100$.

It the case when Panpipe will be the first on the finish line we would earn even $6 \cdot 20 = 120$.

So we can conclude that if the racecourse wouldn’t change the proportions then we could earn there a remarkable amount of money.

Let us leave for a while the racing competitions and apply for another sort of event or to Kangaroo competition.

BREAKTROUGH IV (XXVIII). YOUR INSIGHTS AND ANN’S PENCILS

This is perhaps one of the most subtle problems which were proposed on the Kangaroo competition A.D.2003. Meanwhile in that competition more than 3 millions competitors from 3 continents are participating. In Lithuania the average of participants in the last years stands more or less at 60 000 by the whole population counting 3 700 000. The last statistics was 64 thousands in 6 age groups in 2007.

In Lithuania this problem was proposed in all age groups so it would be right to claim that circa sixty thousands of beautiful minds in Lithuania came in touch with the following task:

Ann has 9 pencils and at least one among them is surely blue. It is also known that:

(A) Among any 5 of her pencils at most 3 different colours appear;

(B) Among any 4 of her pencils at most 3 are of the same colour.

How many blue pencils does Ann have?

We'll refer to *(A)* as to the first Ann's Law and to (*B*) – as to the second Ann's Law.

Let us raise the first fundamental question: how many pencils of different colours may Ann possess?

Could it happen that all these 9 pencils are all of one and the same colour? Could it be two or even three colours? Might it happen that even more than three colours are to be found, say four or more?

If four or more pencils of different colours appear among these 9 pencils, take some 4 pencils representing these 4 different colours and any from remaining pencils as the fifth, and you see that we are contradicting the 1st Ann's Law or statement (A), which was assumed to be right.

So we've proved that all 9 of her pencils can be of at most 3 colours.

So all pencils are either of the same colour or of two or of three colours. No other possibilities are left.

Let us raise now the second fundamental question: how many pencils of one chosen colour may Ann possess?

Again it could be one, two, three, four or more pencils of the same colour. If four or more are of the same colour then take some four of the same colour. Holding them read once the 2nd Ann's Law or statement (B). You see that this is against that law so the described situation is impossible.

Summarizing our efforts we see that we've proved: there are 9 pencils of at most 3 different colours and at most 3 pencils of the same colour. That implies the only possibility left is the possibility, which, expressed in usual words, sounds as follows:

There are exactly three colours with exactly three pencils of each colour.

The answer of the given problem: Ann has 3 blue pencils.

BREAKTHROUGH V (XXIX). ANOTHER KANGAROO PROBLEM, WHICH IS OF GREAT HELP DEALING WITH THE DIFFICULTIES OF EVERYDAY'S LIFE

The real content of that problem is based on the fact that there might be some well-known person from the Lithuanian entertainment industry who once was expected to pay an administrative penalty of approximately 4000 Euro. He brought that amount of money in lorries in the form of the smallest Lithuanian monetary unit – all money amount was in 1 cent coins. For the sake of completeness it could be mentioned that

10 Lithuanian cents correspond more or less to 3 Euro-cents. There was some noise related with this affair in Lithuanian's mass media, and that is exactly what the representatives of that industry always need and welcome.

Let us after that represent that problem as follows:

The famous boss Walliss once fell asleep and had a dream. He dreamed that he is expected to move to his new place of residence. So he would like to take with him all his money, which is exchanged into smallest coins. These coins are all carefully packed and hold in huge boxes. The one of possible explanations of that could be that no thief would be able to steal it. There were 50 such boxes of weights being correspondingly

150, 151, 152,.., 197, 198 and 199 kg.

He decided at once that for the money transport he will involve the transport agency "Quickly moving means safely bringing". He remembered that this agency possesses suitable hermetical lorries carrying 1200 kg each. He started to make thoughts about the least possible number of lorries he should ask for in order to be able to transport all his coins at once.

It should been mentioned that Mr. Walliss was really a bright person. At least since school from which he graduated with gold medal award. So it's no wonder that he started with the estimation of the total weight of all boxes with coins he had. After some seconds he found out that the total weight of all his 50 boxes was expressed by the following sum

$$150 + 151 + 152 + \ldots + 197 + 198 + 199.$$

Mr. Walliss naturally noticed at once that it is possible to rearrange these 50 numbers into 25 pairs with equal sums taking correspondingly one summand from the right and another summand from the left side. The sum in pairs was indeed the same because

$$(150+199)=349=(151+198)=(152+197)=\ldots=(174+175).$$

So he understood that the whole monetary load weighed

$$349 \cdot 25 = \frac{(350-1)\cdot 100}{4} = \frac{35000-100}{4} =$$

$$= 8750 - 25 = 8725 \text{ kg.}$$

Remembering that each lorry takes at most 1200 kg and not more he understood that 7 lorries won't do (clearly $1200 \cdot 7 = 8400 < 8725$). So 7 lorries aren't enough, and he ought to order at least 8 of them. Being logically skilled person he also understood that the inequality

$$1200 \cdot 8 = 9600 > 8725$$

alone doesn't guarantee that the successful loading would be possible and realisable.

He started at once working out the loading plan. The main numbers were 50 (number of boxes) and 8 (needed lorries). Because $50 : 8 > 6$ or, more precisely, $50 = 6 \cdot 8 + 2$, it was clear that in some lorries – at least in two of them – more than 6, that is, at least 7 boxes ought to be loaded. The common sense indicated that these ought to be the easiest boxes. The question remained whether it would be possible to load these 7 easiest boxes in one lorry. Again simple computation demonstrated that then in the first lorry should be loaded at least

$$150 + 151 + 152 + 153 + 154 + 155 + 156 =$$

$$= (150 + 156) + (151 + 155) + (152 + 154) + 153 =$$

$$= 306 + 306 + 306 + 153 = 1071 < 1200$$

and after that 7 easiest of remaining boxes or at least

$$157 + 158 + 159 + 160 + 161 + 162 + 163 =$$
$$= (157 + 163) + (158 + 162) + (159 + 161) + 160 =$$
$$= 320 + 320 + 320 + 160 = 1120 < 1200$$

should be loaded into the second one. This also would fit easily in.

Now there were $50 - 7 \cdot 2 = 50 - 14 = 36$ boxes left and $8 - 2 = 6$ lorries in his disposal, so he could plan to load 6 boxes in each of remaining 6 lorries ($6 \cdot 6 = 36$!).

(**Remark.** *An attentive reader surely noticed that we could keep going on with loading of 7 boxes in one lorry. It would be possible to load 7 successive boxes in the third lorry because*

$$164 + 165 + 166 + 167 + 168 + 169 + 170 =$$
$$= (164 + 170) + (165 + 169) + (166 + 168) + 167 =$$
$$= 334 + 334 + 334 + 167 = 1002 + 167 = 1169 < 1200.$$

But we would fail to load 7 successive boxes in the fourth lorry because the weight we load each time increases by 49 kg what attentive reader had mentioned as well, and now

$$171 + 172 + 173 + 174 + 175 + 176 + 177 =$$
$$= 174 \cdot 7 = 1218 = 1169 + 49 > 1200.)$$

So we load only 6 boxes in each of remaining 6 lorries. In the third lorry we pack the 6 successive boxes of total weight

$$164 + 165 + 166 + 167 + 168 + 169 =$$
$$= (164 + 169) + (165 + 168) + (166 + 167) =$$
$$= 333 + 333 + 333 = 999 < 1200.$$

These 6 boxes fitted easily. Now everything will be completed if we’ll be able to pack 6 heaviest ones in the last lorry. This runs also because

194 + 195 + 196 +197 +198 + 199 =
= 393 + 393 + 393 = 1179 < 1200.

That ends the explanation that Mr. Walliss will be able to carry through his plan and take his 50 boxes with 8 lorries.

He began to feel some pride but in the very moment he get the phone call. His wife was on line. She informed him that she is also informed about the removal of coins and asked him to take also her 4 boxes with smallest coins. The weights of her boxes were correspondingly 200, 201, 202 and 203 kg. We still remember that the heaviest box of her husband weighed 199 kg.

Mr. Walliss remembered that he actually had some weight reserve and became interested whether it would indeed be possible to manage the loading of these 4 additional boxes in these 8 lorries without ordering of additional transport.

His removal scheme as we also remember was the following:

1st lorry takes 150+151+152+153+154+155+156 = = 1071 kg of weight;

2nd lorry takes 157+158+159+160+161+162+163 = = 1120 kg of weight;

3rd lorry takes 164+165+166+167+168+169=999 kg of weight;

4th lorry takes 170+171+172+173+174+175=1035 kg of weight;

5th lorry takes 176+177+178+179+180+181=1071 kg of weight;

6th lorry takes 182+183+184+185+186+187=1107 kg of weight;

7th lorry takes 188+189+190+191+192+193=1143 kg of weight;

8th lorry takes 194+195+196+197+198+199=1179 kg of weight.

And now what to do with these 4 new boxes from the wife's reserve weighing, as it was already mentioned, correspondingly 200, 201, 202 and 203 kg? Were it the only box weighing 200 or 201 kg, then it wouldn't be problem at all – in the third lorry we possess 1200 - 999 = 201 kg of weight reserve. The only box weighing 202 or 203 would already mean some changes in the plan. One such box would cause changes, and Mr. Walliss was expected to deal with 4 new boxes. What should he do?

Firstly of course he made the total weight balance. With four new boxes there come

$$200 + 201 + 202 + 203 = (200 + 203) + (201+202) =$$
$$= 403 + 403 = 806 \text{ kg}$$

of "new" weight. Together with the 150+151+152+…+ +197+198+199=8725 kg of "old" weight it makes a new reality, which weight is $8725+806=9531 < 1200\cdot 8=9600$. But the weight reserve now is really small because this reserve is only 1200 – 1131 = 69 kg. It seems now that successful loading - assuming that it is possible - will be much more complicated than it just had been and Wallis will be forced to take care of each kilogram of weight.

The first remark now could be the following one: no lorry is able to carry more than 7 *boxes because the weight of* 8 *easiest boxes is already*

$$150 + 151 + 152 + 153 + 154 + 155 + 156 + 157 =$$
$$=(150 + 157) + (151 + 156) + (152 + 155) + (153 + 154)=$$
$$= 307 + 307 + 307 + 307 = 1228 > 1200.$$

After that we look with understanding to the following changes in Mr. Walliss plans: he decided to load 12 heaviest boxes in two lorries, 6 boxes in each lorry, and correspondingly pack remaining 54–12=42 boxes in 8–2=6 remaining lorries trying to load 42:6=7 boxes in each of remaining 6 lorries.

He decided also to count carefully how much of weight does he lose as "unused" with each lorry. Recall that his total loses of weight can't exceed 69 kg.

12 heaviest boxes with weights from 192 till 203 can be divided into 2 groups with 6 boxes in each group in the natural way taking successive pairs of remaining easiest and heaviest boxes. This means that in the 1st lorry we are going (or still Mr. Walliss is) to load 6 boxes or 3 pairs with total weight being

$$(192 + 203) + (193 + 202)+ (194 + 201) = \\ = 395 + 395 + 395 = 1185 \text{ kg.}$$

in the first and

$$(195 + 200) + (196 + 199) + (197 + 198) = \\ = 395 + 395 + 395 = 1185 \text{ kg.}$$

in the 2nd lorry.

In each of these cases we've lost 1200–1185=15 kg of weight so our total weight loss is already

$$15 \cdot 2 = 30 \text{ kg}$$

and our reserves for the future loss are 69 – 30 = 39 kg only. Only 39 kg reserve with 42 boxes of coins still to be loaded with weights from 150 till 191 kg which we still hope and believe to load successfully into 6 lorries.

Again Mr. Walliss goes on to packing taking successively one easiest and one heaviest of boxes still unloaded. Speaking more precisely his plan looks exactly as follows:

Still unloaded 42 boxes are located in 7 rows with 6 boxes in each:

150	151	152	153	154	155
156	157	158	159	160	161
162	163	164	165	166	167
168	169	170	171	172	173
174	175	176	177	178	179
180	181	182	183	184	185
186	187	188	189	190	191

and then Mr. Walliss is taking successively 3 remaining easiest and 3 heaviest boxes from both ends and one box or the 7th box from the 4th (middle) row.

These remaining first successive 3 easiest and 3 heaviest boxes have always the same total weight equal

$$(150 + 191) + (151 + 190) + (152 + 189) =$$
$$= (153 + 188) + (154 + 187) + (155 + 186) =$$
$$= (156 + 185) + (157 + 184) + (158 + 183) =$$
$$= (159 + 182) + (160 + 181) + (161 + 180) =$$
$$= (162 + 179) + (163 + 178) + (164 + 177) =$$
$$= (165 + 176) + (166 + 175) + (167 + 174) = 1023 \text{ kg}$$

To these 6 groups of 6 boxes of equal weight we join any of boxes from the (middle) 4th row. Even if we take (or he takes) the heaviest box from this middle 4th row weighing 173 kg, even then we have the total weight of loaded boxes

$$1023 + 173 = 1196 < 1200.$$

This will make only 1200 – 1196 = 4 kg of weight loss in the 3rd lorry. Acting similarly, in the 4th lorry our weight loss will be 1kg greater or 5 kg in total, further on it will be 6 kg in the 5th lorry, 7 kg in the 6th , 8 kg in the 7th and finally 9 kg in the last 8th lorry.

(The balance remains as announced because the sum of these last 6 losses in each of remaining lorries is then 4+5+6+7+8+9=(4+9)+(5+8)+(6+7)=13+13+13=39 just as expected.)

BREAKTHROUGH VI (XXX). PSYCHOLOGICAL DANGERS OF SIMPLEST FORMULATIONS

They gazed in delight, while the Butcher exclaimed,
"He was always a desperate wag!"
They beheld him – their Baker – their hero unnamed –
On the top of a neighbouring crag.

Rather often telling the accidents or citing exciting histories of everyday-life or science we summarize making conclusions of the kind:

The main hero looked so honestly but it turned out that he was just as weak as our average neighbour is;

or

The circumstances seemed to be so uncomplicated but all his efforts leaded him to nothing;

or

He was hopeless in start but strong as a lion at finish;

or

Even I couldn't achieve more in this whole affair.

Solving mathematical problems we enjoy plenty situations of similar kind with the difference only that in mathematical world and galaxy the things are developing in more "cultural" or subtle or at least in not so straightforward way. Saying that, first of all we have in mind and never forget that in mathematical galaxy, if you've great problems or difficulties, you may quietly leave everything aside for a while, take your time and then come back to these matters with a new mind resources and fresh head and again try to achieve more than it was possible to achieve.

We only would like to draw your attention to the fact that even you and me may come across the problems which even the brightest heads of the mankind were solving thousands of years. Solving but still unsuccessfully. You never know. You ought to be prepared to all possibilities. Life is so rich on possibilities of every kind, size and calibre.

The seriousness of the situation could be and is nicely described in the classical verse due to Alfred Edward Housman (1859 -1936):

The Grizzly Bear is huge and wild,
He has devoured the infant child.
The infant child is not aware
He has been eaten by the bear.

So it can happen that the proposed problem is understandable, its formulation is so short that sometimes everything – all conditions and tasks – is packed in just one sentence, but at that place no achievements are possible, no considerable results are to be expected and no hopes are left. Nothing is possible for me, also for you, also for both of us and even for all three of us as well. What is even worse on that stage – and it happens unfortunately rather often – that you and me, we have already promised to our neighbours and relatives to do this problem, to finish all that in an hour or during one day, or in two weeks, or in three months. We promised already that in this year A.D. 2007 everything will be done, completed, everything until least details will be finished and forgotten. It couldn’t be otherwise, there is no other choice because you and me, because both of us are so bright, so smart and, by the way, all circumstances are so promising.

Our psychological guide or everyday experience ought to remind us every day that the problems we might happen to meet may be and are so different. Some or even most of them are really easy, but some might happen and are more serious but still accessible. There are also perhaps quite few of them which are indeed extremely complicated. And there are some which are impossible. They were impossible. They are impossible. It might happen that they will always remain impossible. For all times and age groups.

Our human experience obliges us not to separate ourselves from such most complicated or simply impossible problems alone for the reason that they are also part of our human landscape and life.

We do not intend in a slightest degree to frighten our possible readers; we do not want also to reduce the courage of our readers. Still the importance of being honest demands that we are expected not to lose the ground and remain realistic. At least for the sake of completeness we must stay, state and repeat that such situations are possible also in our life and in our days. We must keep that all clearly in our minds.

We remind that the person who wants to learn to think logically must be prepared to regard every possible situation with the circumstances of every kind.

As already was told and repeated - we do not intend to hide anything we know and understand from our readers. Some situation for any courageous person can be inspiring enough to try to manage the situation which for many if not for all is not possible to manage.

It's so human and understandable. It is so attractive and difficult. But it's your choice and responsibility.

And it's also your joy and pleasure.

IT'S EVEN UNNECESSARY TO COME ACROSS THE FERMAT'S PROBLEM

Let us cite the remarkable aphorism of H.Steinhaus:

Pythagoras wasn't an Englishman. If he was an Englishman then the famous statement would be formulated as follows: I think that the square on the hypotenuse….

More that 300 years a standard example of inaccessible problem, which could be formulated in one sentence, was the famous Fermat's problem. This problem ruined the life of many persons, who believed they are obliged to solve it. They became somehow convinced that they necessarily must do it.

We will recall its formulation for the sake of complicity using more than one sentence. Nevertheless it could be also regarded to be "one sentence problem".

Firstly we notice that it is possible to find two squares of integers which sum is also a square of an integer.

First example of such a kind is most famous right triangle with integer sides 3, 4 and 5, because

$$3^2 + 4^2 = 5^2.$$

Applying the natural art of making analogies we'll easily come to the following reflection: seeing that there are (a lot of) two such squares of positive integers whose sum is also the square of a positive integer it is so natural to ask whether the same might happen also with two cubes of positive integers or to ask:

Are there indeed two such cubes of positive integers such that their sum is also a cube of a positive integer?

The same question could be repeated for the sum of fourth, fifth and *n*-th degrees of two positive integers.

In that way the Fermat's theorem came into being in 17^{th} century and turned out to be very capturing problem. This is also due to the shortness of formulation. Short questions are natural questions to ask. So is should be repeated that **shortness can be capturing; that's why it is extremely involving and dangerous.**

We could try to reformulate it in one sentence in a somehow funnier version:

Even in science things, which are created using one sentence only, may cause much harm.

In the history of mathematics of the last centuries the name of probably most capturing sentence was Fermat's last theorem.

Do there exist such three positive integers x, y and z that x to power n plus y to power n is equal z to power n or,

$$x^n + y^n = z^n \ ?$$

From the history of mathematics we know that considerably big amount of money was proposed as an award for successful solution of that problem. Many thousands of solvers were fighting for success on that area hoping finally to find such three integer numbers x, y and z for some $n > 2$.

More than three last centuries no one, who was dealing with that problem, could repeat *“Eureka”* after Archimedes. In fact many thought they did prove it. Bur after careful examination of their arguments it always appeared that the problem still remains unsolved. Dealing with the ideas, which were awakened by attempts of proving Fermat’s last theorem, many new and beautiful mathematical theories were being created. The theorem became so famous that it was compared with the hen, which carries golden eggs.

One shouldn’t forget that the efforts in that direction were stimulated by the famous words of Fermat who claimed he knew the proof only the margins of the book where it had been written were too narrow for writing down the proof.

He stated that „It is impossible for a cube to be the sum of two cubes, a fourth power to be the sum of two fourth powers, or in general for any number that is a power greater than the second to be the sum of two like powers. I have discovered a truly marvelous demonstration of this proposition that this margin is too narrow to contain."

The authority of Fermat was so big that many investigators were so eager to find out what he could have had in his mind. This lasted for centuries.

Only in the very end of 20th century it appeared that there are no such three numbers *x*, *y* and *z* in the case when $n > 2$.

It was extremely remarkable event in the well-organized and stable mathematical world. From the psychological point of view the most exciting matter was probably the following one. The final part of the proof contains about hundred pages, which are fully understandable and accessible only to, say, 10 top mathematicians of nowadays. (*This is a subjective view of the author. – Editor*.)

Let us recall that at the beginning there was an almost ordinary looking question formulated in one sentence.

PROBLEMS WORTH MILLION DOLLARS EACH

Idea to stimulate the progress proposing money for successful efforts is as old as mankind itself. In Internet you find some problems for solving of which million dollars are offered. For example: Two publishers are offering million dollars to anyone who can prove that each even number is a sum of two prime numbers.

All you need to know there is what prime numbers are and what is the sum of integers. Absolutely nothing more. You hardly can find a person, which wouldn't know it. It's really difficult to find a person who didn't hear at least 20 times that a prime number is a number greater than 1 which has no other divisors different from 1 and itself.

We would like once again to draw the attention of the reader, we would like remind you again and again that there were, there are and there always will be difficult problems in the world. There are a lot of such problems. There are plenty nice problems which were treated by many bright minds. Were treated but remained unsolved. So remember that it can happen that even you today and tomorrow are not and will not be able to solve it. Don't be angry that you can't solve it. There are so many problems you can solve. There so many problems which are accessive for you and can remarkably help you to encrease the deepness of your mind.

Only don't be afraid. In no case. You can much more than you usually believe you can.

BREAKTHROUGH VII (XXXI). AGAIN AN APPARENTLY SIMPLY LOOKING PROBLEM

Let us take any positive integer or any natural number you wish and let us arrange with that number the following procedure: if we happened to come across an even number we'll divide it by 2. In the opposite case, that is, if we chanced to meet an odd number then we'll multiply it by 3 and add 1. With either number we got we'll repeat that procedure and will proceed trying to establish what will happen?

Of course we could and even should to describe it by the formula. It would take less place and alone from that reason such a description is worth mentioning and applying.

In other words, if the number N is even, then $f(N) = N/2$ and if N is odd, then $f(N) = 3N + 1$.

As a natural example of that kind we will take the number of the year and will try to establish what could be expected after few steps.

Because the number of the year or 2007 is an odd number so we'll go to the thrice bigger number and add 1. So we will get

$$3 \cdot 2007 + 1 = 6022.$$

6022 is clearly an even number so in the next step we are to divide it by two getting $6022 : 2 = 3011$. There is no doubt that 3011 is again odd integer, so we intend again to multiply it by 3 and add 1 or to go over to the number $3 \cdot 3011 + 1 = 9034$. This number is again even, so in the next step we'll get $9034 : 2 = 4517$. Continuing we will get (dear reader, wouldn't you be so kind as to assist our steps with the functioning calculator in hand, please!)

13552, 6776, 3388 1694, 847 (at least now we've stopped dividing by 2). Further we will get 2542, 1271, 3814, 1907 (we are in the neighbourhod of our start number 2007, difference is exactly 100). The process runs further bringing us to the numbers 5722, 2861, 8584, 4292, 2146 (again we are not very far from 2007).

Further we will see 1073, 3220, 1610, 805, which are followed by 2416. This number 2416 initiates a rather long way down giving numbers 1208, 604, 302, 151. Further „intermediate stations" are 454, 227, 682, 341, 1024. Now it has happened that we came across the power of 2, so we will go down as deep as possible and reach 1. Indeed 1024 is followed by 512, then 256, 128, 64, 32, 16, 8, 4, 2, 1.

We've got the smallest possible positive integer or 1. Of course, by inertia we could apply our procedure further but then we would be moving in circle $1 \to 4 \to 2 \to 1 \to 4 \to 2 \to 1$ and so on without any end.

Let us take another integer, for example, the greatest 4-digital number 9999 and see whether we again (if ever) will reach the smallest possible integer or 1.

Let's start and announce some kind of philosophy.

We do not know whether our road will be long. We do not know even whether it would lead us to 1. But it ought to be stated and repeated that comparing it with our first time experience which we gathered working with the number 2007 we'll learn a lot of arithmetical life following that iterated procedure. We do not intend to contradict the Latin phrase

Testis unus testis nullus.

It's in Latin and originally it expresses that you can't know (all) the truth about something with just one witness. From the lawyer's point of view you can't really prove anything what you are eager to prove having only one witness.

But to prove is something different from to learn and we are not in court – we would like to feel and to learn about how the things got arranged.

So denoting the procedure of going over to the following number by arrow or → we will consequently get

9999 → 29998 → 14999 → 44998 (we have already almost 5 times as much as we had when we started) → 22499 → 67498 (new record of magnitude) → 33749 → 101248 (wow, we are actually 6-digital!)→ 50624 → 25312 → 12656 → 6328 (only 4 digits left, and that's not the end; we are still falling down!) 3164 → 1582 → 791 (happily odd number again) 2374 → 1187 → 3562 → 1781 → 5344 → 2672 → 1336 (again it's going down) → 668 → 334 → 167 → 502 → 251 → 754 → 377 → 1132 (up and down)→ 566 → 283 → 850 (do you still remember how high we'd been after the start?) → 425 → 1276 → 638 → 319 → 958 (again fast thousand) → 479 → 1438 → 719 → 2158. The number 2158 is again in the neighbourhood of a number of a year or 2007, and that number 2158 is subsequently followed by 1079 → 3238 → 1619 → 4858 → 2429 → 7288 (now we'll be again several times dividing by 2) → 3644 → 1822 → 911 → 2734 → 1367 → 4102 (merry goes round) → 2051 → 6154 → 3077 → 9232 (we are in the near of our initial 9999) $\rightarrow$ 4616 $\rightarrow$ 2308 $\rightarrow$ 1154 $\rightarrow$ 577 $\rightarrow$ 1732 $\rightarrow$ 866 $\rightarrow$ 433 $\rightarrow$ 1300 (for the first time number with two zeros)$\rightarrow$ 650 $\rightarrow$ 325 $\rightarrow$ 976 $\rightarrow$ 488 $\rightarrow$ 244 $\rightarrow$ 122 $\rightarrow$ 61 $\rightarrow$ 184 $\rightarrow$ 92 $\rightarrow$ 46 $\rightarrow$ 23 $\rightarrow$ 70 $\rightarrow$ 35 $\rightarrow$ 106 $\rightarrow$ 53 $\rightarrow$ 160 $\rightarrow$ 80 $\rightarrow$ 40 $\rightarrow$ 20 $\rightarrow$10$\rightarrow$ 5 $\rightarrow$ 16 (thanks God) $\rightarrow$ 8 $\rightarrow$ 4 $\rightarrow$ 2 $\rightarrow$ 1.

We could now declare with some pride that we are done or that we are on the end of our road. Our practical experience has increased and is inspiring us to formulate the following natural question.

Starting from any positive natural number N and applying our procedure: going over from N to N/2 if that N is even and going over from N to 3N +1 if that N is odd, and repeating that procedure for sufficiently long, would we always be able to reach the number 1?

Now we will present another problem together with the question: which of both seems to the reader to be more difficult and complicated?

So we ask you very seriously to decide which of these two questions appears to you to be more inaccessible?

We'll present that second problem with slight adoption. Recall the tale about two well-known world-wide famous young persons whose names are *DAPHNIS* and *CHLOE*. Let us say some words about the continuation of their exciting and sad history.

After their death which we will denote as a transition to the better world they both learned that this better world consists of infinitely many disjoint spheres S_n *enumerated with natural numbers n. Both may land on any sphere* S_m *and it is clear for us that in any case they will go on trying to find each other. There are two types* *of transition between these spheres which could be described as a two-sided traffic between them:*

*(A) From any sphere with the number n it is possible to go to the sphere with the number 2n and vice versa **(transition of the first kind**);*

*(B) From any sphere with the number n it is possible to go to the sphere with the number 3n + 1 and vice versa **(transition of the second kind**);*

If you believe that DAPHNIS and CHLOE are able to find each other then please explain what could they undertake for that?

Needless to explain that "to find each other" in that context means for them both to land on the same sphere S_n *and to be aware of it.*

Is it really possible and what could you advice for that?

In order to gain some experience imagine that *Daphnis* is actually on the sphere with the number 2007 and ask whether he could reach the sphere with number 1 from that sphere with number 2007.

Where can he go from the sphere with number 2007? This number is odd so he can't apply transition of the first kind, so it remains to apply the transition of the second kind and to go to the sphere with number $3 \cdot 2007 + 1 = 6022$. Now he can move to the sphere with number 3011. But *Daphnis* didn't do so. He went to the sphere with number 12044, again applying the transition of the first kind. Why did he do it? He could go to the sphere with number 3011. He didn't. Why? Recall that in his dreams he is already on the sphere with number 1.

Now we will seldom comment his moves but we'll always thouroughly indicate where is he actually. Recall once again – his clear aim is to land as soon as possible on the sphere with number 1, and for that he is equipped with the transitions of the first and of the second kind or (two-sided) transitions $n \leftrightarrow 2n$ and $n \leftrightarrow 3n+1$.

From the sphere with number 12044 Daphnis had chosen the sphere with number 24088 (curious step, increasing instead of decreasing), then because $24088 = 3 \cdot 8029 + 1$ he went to the sphere with number 8029.

Further way will be indicated with two-sided arrows:

$8029=3\cdot2676+1 \leftrightarrow 2676 \leftrightarrow 1338 \leftrightarrow 669 \leftrightarrow 2008 \leftrightarrow 1004 \leftrightarrow 502 \leftrightarrow 251 \leftrightarrow 754 \leftrightarrow 377 \leftrightarrow 1132 \leftrightarrow 566 \leftrightarrow 283=3\cdot94+1 \leftrightarrow 94=3\cdot31+1 \leftrightarrow 31 \leftrightarrow 10 \leftrightarrow 5 \leftrightarrow 16 \leftrightarrow 8 \leftrightarrow 4 \leftrightarrow 2 \leftrightarrow 1$.

Are you now able to see how succesfull he proved himself to be? Are you able now to explain each step of *Daphnis* route?

There are some really understandable things.

1. Because all transitions are two-sided it means that if Daphnis, being on the sphere 2007, can reach the sphere 1 then he is also able to return back from the sphere 1 to the sphere 2007. That is indeed so because each step is either transition of the first or of the second kind and is two-sided (we remind that for these steps we started the use of two-sided arrows).

2. If from any sphere N you can reach the sphere 1, then from any sphere N you can also reach any other sphere M. Indeed, if, starting from the sphere N or from the sphere M you can reach the sphere 1, then, using *two-sidedness, you can reach the sphere M from the sphere N.*

So now the fundamental question whether from any sphere *N* you can reach any other sphere *M* is reduced to the question whether from each sphere *S* you can reach the sphere 1.

We may assume that under such sad separation of these loving hearts they both are already much more skilled and, indeed, if they do not know on which sphere they could find each other, then after some consideration they both independently ought to be able to understand that they must make a transit as soon as possible namely to the sphere number 1.

So now, dear reader, we repeat our question:

Which of these two problems according to your opinion and mind appear to you to be more difficult?

We remind you both of them once again:

Problem 1. Given any positive integer N. If N is even then go to the number N/2, and, if N is odd then go to the number 3N + 1. Do we always get 1, applying that procedure repeatedly?

Problem 2. Having any positive integer N you can go to either 2N or 3N + 1 or vice versa. Are you always able to get 1, starting from any integer N and applying that procedure repeatedly?

Which of these tasks appears to be more difficult to your fantasy?

One of the problems just mentioned is still unsolved (usually they call it hypothesis) decorated already with three names of prominent mathematicians having dealt with it, and the second is the problem 9 from the mathematical Baltic Way team contest A.D.1997 in

Copenhagen (vide [1]or [3]) The solution of it will be presented some lines below.

So who is who? What is normal and what is difficult? What is possible and what is not? What can be achieved in an hour and what can't be made in 20 years?

Don't you find that the decision woudn't be easy?

Once again we came across to the well-known wisdom claiming that

APPEARANCES ARE DECEPTIVE

It is related to the statement that you meet the person according to his dressing and bid farewell already according to his mind.

From the (psycho)logical point of view nothing very special – you are getting inside from the outside and seldom in using other ways.

Concerning mathematical adventures you could also notice that you meet the problems according the shortness of their formulation (formulation is used to play the role of dressing) and bid farewell already according to the accessibility of solution.

At this place we would like again to cite the famous Belorussian composer of mathematical problems (and a brilliant solver of them also!) Sergey Alekseewich MAZANIK who is a professor at Minsk University having noticed once:

The problem in number theory with formulation length of one sentence is usually deadly difficult for the solver.

The answer in our case about what is easy and what is not is the following one:

The problem 2 is accessible.

The problem 1 REMAINS DEADLY DIFFICULT (and the day of its possible solution hasn't come yet).

We would like also, frankly and openly, with the considerable dose of normal optimism, remind the reader that there is nothing more valuable in life than that. On the other hand it is indeed possible that on the very day when that problem will be done it will appear that the solution is so simple that the problem itself wouldn't regarded as a difficult task.

But not earlier than the short solution would appear.

We will carefully regard what did *Daphnis* intend, to disclose the wisdom which lead him.We must carefully analyse what we've seen because we intend also to prepare the instruction for the future solvers in the form of simple advices which he could apply in every case.

The first fundamental wisdom or advice for transitors of spheres is addressed to those who intend to reach sphere 1 from any other sphere N:

You will prove yourself to be successful if you are able to prove that starting from any sphere N you are **always** able to reach **some** other sphere K with $K < N$, if only $N > 1$.

It is enough to prove that you ALWAYS are able to make at least one step down.

Because if you are able to make at least one step down then repeating it you will always reach the sphere 1.

Let's start preparing that instruction.

This will be an instruction containing three steps or advices which will always work.

1st advice how to reach a lower sphere is adressed to those who actually are on the spheres with numbers

1, 4, 7, 10, 13, 16, 19, ..., 100, ..., 2005, ...

or in general case on the spheres $3N + 1$.

Second advice is firstly meant for those who actually are on the spheres

2, 4, 8, 11, 14, 17, 20,..., 200,..., 2006, …

or in general case on the spheres $3N + 2$.

Third advice will be for those who are on a sphere whose number divides into 3 or on the spheres

3, 6, 9, 12, 15, ..., 300, ..., 2007, …

or in general case on the spheres $3N$.

Don't forget that our aim is to go down in any case.

In the first case this is done in one step using the **transition of the second kind** allowing us to go from the sphere $3N + 1$ to the sphere N and mentioning that, of course,

$$N < 3N + 1,$$

and so *Daphnis* would get lower than he'd just been.

In the second case our advice will consist of two steps: first we are going up using ***the transition of the first kind*** and going from the sphere $3N + 2$ to the sphere $6N + 4$. Then using **the transition of the second kind** we'll go from the sphere

$$6N + 4 = 3(2N + 1) + 1$$

to the sphere

$$2N + 1 < 3N + 1$$

and that means that we get lower again.

In the third case our advices will be most complicated of all what we've seen and will consist of several steps.

Firstly we'll be unexpectedly many times going „up". Our first move from sphere $3N$ using the **transition of the second kind** will be to the sphere $3(3N)+1=9N+1$.

Then we'll two times apply **the transition of the first kind**. So from $9N+1$ it leads to $2(9N+1)=18N+2$, then from $18N+2$ it goes to $36N+4$. We are now more than 12 times higher than we'd been. Now it's going down, because mentioning that $36N+4=3(12N+1)+1$ Daphnis has transit rights to reach $12N+1$. Mentioning that $12N+1=3(4N)+1$ he has rights together with us to go to $4N$. These were two applications of **the transition of the second kind.** Now, of course, he'll use the transition of the first kind and will reach the sphere $2N$. And, of course,

$$2N < 3N$$

so that we all are again lower than we'd been.

So in each case we all (and Daphnis with or without us) are able to make a step down. Combining several such steps we are always able to reach the sphere 1.

And concerning the problem 1, which carries the names of three brilliant mathematicians, the only thing that seems to be known is that we can prove that on some step we will get the number which is divisible by 4, not just by 2.

That seems to be exactly the whole progress in that direction.

This is not a problem for solving of which millions are suggested, but nevertheless something of the kind.

So similar problems might have completely different fates.

Let us for fun cite again some lines of the famous poem „Hunting of the Snark“

They roused him with muffins- they roused him with ice-
They roused him with mustard and cress-

They roused him with jam and judicious advice –
They set him conundrums to guess.

For the sake of complicity let us add that conundrum is a riddle, especially one with a pun in its answer.

The author dares not to express his bold opinion that Lewis Carroll, writing his poem in our days, would formulate the last line as

„They set him sudoku to guess."

BREAKTHROUGH VII (XXXII). AGAIN DISCUSSING WHAT TO DO WHEN IT IS NOT AT ALL CLEAR WHAT

The title of this chapter is somewhat paradoxal and slightly contraversial but even in the case when we do not know what to do it is possible to undertake some clever moves.

It reminds a Lithuanian fairy-tale telling us about a bright boy and how having nothing but an axe he'd boiled a tasty meal.

We are going to present to you something very similar.

A.D. 2000 in Lithuania in the regional mathematical olympiad the following problem was proposed. This problem in turn was an adoption of some problem which in the same Bimillenium year was proposed on the final stage of the Lithuanian students' mathematical olympiad.

I've already written about that problem (vide [1]) and about some rather interesting psychological circumstances which are always connected with non-standard matters.

Let us firstly present the problem itself. Formally and frankly speaking this is not at all the high-school problem because an equation of the third degree with three variables *x*, *y* and *z* appears in it, making the whole thing to be not quite the high-school subject.

But never mind: perhaps right now you are in that state of mind when

YOU CAN MORE THAN YOU EVER SUPPOSED.

We consider the equation

$$x^2 + y^2 + z^2 + 10 = xyz$$

in positive integers x, y and z.

It means that first of all we are eagerly looking for some three positive integers in order to replace *x*, *y* and *z* and get the right equality.

Let's continue with the formulation of the problem.

(A) *Present us one such triple (x, y, z).*

(B) *Find 7 such triples.*

(C) *Does that equation possess 2000 solutions (today we would ask about 2007 possible solutions)?*

(D) *Does that equation possess infinitely many solutions?*

"Infinitely many solutions" means that for any given number *N* (which might appear to be unreally big) you and me or we all are able to find at least N of such suitable triples or at least prove that such triples exist.

Our aim is to structure the problem. So part *(A)* in which we ask to look for one solution consisting of 3 positive integers is practically at the same time an invitation to solving. Without saying any word we are clearly giving to understand that it is not difficult to find a solution. Look and find, say it to us, we are waiting. We know that it won't take a long time.

We mentioned already that asking for something which is obvious is the way to involve us into action. We become interested and we are ready for further efforts in that area where we've started so succesfully.

So in the beginning there was a (simple) question: show us three suitable integers x, y and z satisfying the equation.

Part *(B)* asks already for (at least) 7 such triples.

Part *(C)* wants to be sure about thousands of possible solutions (appetites usually are growing faster than abilities or food supplies).

Part *(D)* is already almost philosophical and is practically asking about possibilities to ensure that there is an unbounded amount of solutions.

In fact to be unbounded means – *say us any (big) number N you wish and we will demonstrate that there are at least N of such solutions*.

After that regional Olympiad I mentioned that problem in an article which appeared in some Lithuanian computer magazine. Later I've got a letter from the reader. The author of the letter was an associate professor for informatics at University.

In that letter some extremely natural surprise connected with the discussed problem was expressed in words:

I took the look to that problem and I didn't know what to do.

Other thoughts which weren't expressed by words and remained below the water might be (and almost for sure were) the following ones:

How can it happen that I am not able to solve a problem which was proposed for the students?

The question is frank, serious and very natural. It is human question.

Let us again regard the question: is that problem really difficult?

The letter of the reader indicates that the question is well exposed. If at the first glimpse it is not quite clear what to do it means that the question is suitable and may be useful.

Part *(A)* asks really, as it was already told, for something almost trivial: to find or to guess one suitable triple for our equation – no formulas need to be mentioned or calculations applied. Find out and say, nothing more. Making an effort you'll gain an experience. Unavoidably. You'll become cleverer. You'll see right now more than just before.

At that place it could be frankly exposed that our attitude towards the guessing and forecasting especially at school is cautious. We understand such teachers and share their views perfectly well. No one is going to deny the value of foreseeing only because sometimes there are some doubts concerning the sources of such knowledge. The origin of that sources may be based on the mind power of my neighbour or on inscriptions in his exercise-book. Could be and not seldom are. So the fact that we are taking that sudden knowledge with some reserve is understandable but on the other hand it doesn't deny or contradict even in slightest degree the undoubted value of such foreseing.

Needless to remind that not in vain all TV news begin or end, or even both, namely with weather forecast and we are listening to it with all our seriousness. This has also very much in common with the art of prognostic and prediction and with other notable sciences and skill resources.

The author had been discussing exactly that problem and such situation in the very different audiences from youngest students untill these of skilled teachers. I must state that at least after 20 seconds someone from the audience, when asked, was able to present the example of such concrete triple *x*, *y* and *z* satisfying the given equation.

I would like to add that such wanted triple was always the triple (3, 4, 5). This triple indeed is a suitable one because plunging it in the given equation we get

$$3^2 + 4^2 + 5^2 + 10 = 3 \cdot 4 \cdot 5$$

because

$$9 + 16 + 25 + 10 = 60 = 3 \cdot 4 \cdot 5.$$

Stating that, we are done with the part *(A)* and become much more involved and interested in the whole process of solving, and we’d like to ask first of all:

What about the possible obstacles when completing part *(B)*?

The next task or part *(B)* was the question about 7 suitable triples. Regarding that first solution or most famous one from all Pythagorean triples

$$(3, 4, 5)$$

we gathered also some precious insights concerning the question “how is that equation made and constructed?”.

How is it arranged? First of all it ought to be mentioned that our equation is arranged in so-called democratical way or in such a way that all variables or unknowns “possess equal rights” or are participating in that equation "all in the same way". The influence of every unknown magnitude is the same as the influence of the other unknown magnitude. No one is participating differently than the other and no one is more outstanding.

That kind of democracy when all variables are involved in the same way has its special name: such equations are being called symmetrical equations.

What does it practically mean and provide for the world of possible solutions?

In that world this means the following: having one solution (founding as in our case the triple (3, 4, 5)) we could then change places of these positive integers in that triple getting again a solution. Otherwise if it were not so then the participation of variables in equation wouldn't be the same or symmetrical. By the way also in mathematics some kind of partial democracy in changing places is also possible, e.g. just as already mentioned symmetry.

So given a suitable or "good" triple such as (3, 4, 5) we can change places of variables in arbitrary way. We'll successfully proceed in getting suitable or „good" triples. So from (3, 4, 5) we would get more suitable triples or (3, 5, 4), (4, 3, 5), (4, 5, 3), (5, 3, 4) and (5, 4, 3).

Together with initial (3, 4, 5) we have then already 6 solutions.

If we were able to find another solution, the 7^{th} one, we would also be done with part *(B)*.

We can state now that two so simple things like a foreseeing of one solution + mentioning the democracy of equation brought us fast 2 of 4 parts of the solution.

“The rest of my speech” (he explained to his man)
“You shall hear when I’ve leisure to speak it.
But the Snark is at hand, let me tell you again!
‘Tis our glorious duty to seek it!

Now we need some additional impulse for the final move. You’ll be astonished hearing that this additional impulse will be connected with the most usual idea of quadratic equation.

We see that the original equation contains three variables *x*, *y* and *z*; every variable may take different values and the fragment of equation containing their product *xyz* indicates that is not quadratic, that is already a cubic equation. So this is not a school product, but nevertheless…

This is indeed not a school subject but still in 2 seconds it may be reduced to something that is completely school product or at least more school product and subject than anything else.

Let us take a glimpse of what now is going to happen.

We will play such a game: this original equation is not quadratic but if we were plunging in it 4 instead of *y* and 5 instead of *z* and will go on pretending that we’d forget the value of *x* (in the reality it is not necessarily so! Hush, we remember that *x* was something similar to 3, but we are continuing with our game hoping that it would lead us somewhere).

What are the possible profits of our short memory or from all this theatre claiming that we’ve completely forgot that value of x? It would lead us to a banal quadratic equation. And what would arise from that?

Another value of x will arise from that. Quadratic equations are equations with **two** solutions. And exactly this makes the whole sense and profit of our performance. This second value of x. Thank you very much! This might be useful. Let's enjoy further details.

So plunging 4 instead of y and 5 instead of z we would get the equation

$$x^2 + 4^2 + 5^2 + 10 = x \cdot 4 \cdot 5$$

or

$$x^2 - 20x + 51 = 0.$$

This equation really possess also another solution different from 3 and that different solution is $x = 17$.

This $x = 17$, being another solution of that quadratic equation, leads also to the triple (17, 4, 5) which is the new solution of our initial equation.

Formally speaking this is already the 7^{th} solution we're asking for, and this completes the part *(B)*.

Afterwards the permutations of that triple (17, 4, 5) may and will follow. These permutations we are allowed because of democratic structure or symmetricity of our equation.

From the triple (17, 4, 5), changing places of the participating numbers, we'll get also triplets (17, 5, 4), (4, 17, 5), (4, 5, 17), (5, 17, 4) and (5, 4, 17).

We have got already 6 + 6 or 12 solutions. In fact we've learned also how to produce new solutions from the so-called old ones. Again we could take the triple (4, 5, 17) and plung into initial equation 5 instead of y and 17 instead of z and will once again go on pretending that we have just forgot the value of x.

(Again, dear reader, don't worry, you and I or we all remember that x could indeed be 4!)

Repeating that game and process we get the quadratic equation

$$x^2 + 5^2 + 17^2 + 10 = x \cdot 5 \cdot 17$$

or explicitely

$$x^2 - 85x + 324 = 0,$$

which leads us to other solution or to $x = 81$. So (81, 5, 17) is already the 13th solution and this process may be continued without any end (why are we so sure that we won't one day be moving in a circle?).

For the careful reader we could advice right now to consult the Vieta's theorem which connect the roots of quadratic equation with its coefficients and see how it ensures that our process of pretending will at every step produce a new solution of quadratic equation and consequently provide new triples satisfying our equation.

Indeed, everything we need is to be sure that the following is always right.

Assume that $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ *is a solution of the equation* $x^2 + y^2 + z^2 + 10 = xyz$ *such that* $\mathbf{x} < \mathbf{y} < \mathbf{z}$ *Denote the sum of variables* $\mathbf{x} + \mathbf{y} + \mathbf{z}$ *by K.*

Then the triple $(\mathbf{yz - x}, \mathbf{y}, \mathbf{z})$ *is another solution of the given equation with greater sum of variables K.*

Proof. Indeed if $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ *is a solution of the equation* $x^2 + y^2 + z^2 + 10 = xyz$*, then*

$$\mathbf{x^2 + y^2 + z^2 + 10 = xyz.}$$

All we need now is to prove that

$$\mathbf{(yz - x)^2 + y^2 + z^2 + 10 = (yz - x)yz.}$$

This is the same as

$$\mathbf{y^2z^2 - 2xyz + x^2 + y^2 + z^2 + 10 = y^2z^2 - xyz.}$$

Subtracting $\mathbf{y^2z^2}$ *from both sides and carrying* $\mathbf{-2xyz}$ *to other side we get the initial equality*

$$\mathbf{x^2 + y^2 + z^2 + 10 = xyz}$$

which is fulfilled by initial assumption.

It remains to be noticed that the condition $\boldsymbol{x} < \boldsymbol{y} < \boldsymbol{z}$ *guarantees that the new sum*

$$K = (\boldsymbol{yz} - \boldsymbol{x}) + \boldsymbol{y} + \boldsymbol{z}$$

of variables is greater than the "old" $K = \boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z}$ *because indeed* $\boldsymbol{yz} - \boldsymbol{x} > \boldsymbol{x}$.

We would like to ask the reader (using the same ideas) to solve the original equation which, as it was mentioned already, had been proposed in the Lithuanian School Olympiad, vide book of Mačys [4], p. 36.

The equation

$$x^2 + y^2 + z^2 + u^2 = xyzu$$

is to be solved in natural numbers x, y, z and u.

(i) *Find at least one solution;*
(ii) *Find at least 33 solutions;*
(iii) *Prove that there are at least 2000 solutions.*

BREAKTHROUGH IX (XXXIII). THAT ALLMIGHTY IDEA OF SIMPLIFICATION

Taking three as a subject to reason about –
A convenient number to state
We add Seven, then Ten, and then multiply out
By One Thousand diminished by Eight.

It sometimes happens that we do not regard that „allmighty idea of simplification" as something being of extreme importance, though theoretically and ideologically we are completely aware that it is the universal method.

Let us take a look at the problem which we will try to reformulate more than it has been already reformulated. Be more adopted than we've seen it to be. The initial adoption is again due to the famous Sankt-Petersburg composers of problems for beautiful minds (vide [5], problems 29 and 35, p. 14 and 15).

Adoption is very important. In ideal case it means that the problem is presented with such details that we tend to believe: everything described in the text has actually happened in the real life. We'll never give up any efforts to decorate the problem with attractive and meaningful circumstances which are more than understandable for our possible reader.

This is not easy to achieve but it is always worth doing.

At this place we would like ask you to estimate what form of the problem you'd prefer:

(*A*) **strict and formal** without any unnecessary words

or

(*B*) with some elements of **decoration, application and adoption.**

Versions which are more similar to (*A*) we will call **normal versions**, and those more similar to (*B*) we will refer to as **human versions.**

Both forms, either **normal** or **human**, have **clear advantages** of their own.

1. Normal *(strict or, so to say, more scientific) version.*

Good balances without weights and 9 closed vessels weighing 1, 2, 3, 4, 5, 6, 7, 8, 9 kg are given. The weight of each vessel is being indicated on it. Into one of these vessels 1kg weight was put in.

Making two weightings determine into which vessel that weight was put in. Is it possible?

2. Human *(popular and more appealing) version which we'll actually risk to present with fragments of possible adventures.*

In ideal case and form it could be as nice as a fairy-tale.

Winnie-the-Pooh is sitting on the strongest horizontal branch in the oldest oak in the huge forest and dreaming as always about non-stopped honey consuming. The honey supplies for that period of time were satisfactory sufficient. Next to him on the same branch 9 vessels with honey weighing correspondingly

1, 2, 3, 4, 5, 6, 7, 8, 9

kilograms were carefully and safety located and on each vessel its weight was clearly indicated. He intended actually to start the consuming process but was suddenly interrupted by the visit of conservationist Big Cat. Big Cat was the major of the forest. Big Cat once again carefully examined all licensies of Winnie: first of all the permission to habitate in the forest on the branch, then the licence to work with balance and the permission to keep the food supplies, especially these with honey in vessels.

All permissions and licencies were O.K., still some tension remained in the air because it was a public secret even for Winnie that the Big Cat wanted to move him out from that oak. He visited him practically every day, often repeating to Winnie that other inhabitants of the forest complained to him about Winnie's attitude and behaviour. Main reasons for the complaints were the following: the branch cracks are making too much noise, he snores also too loud and he mouthes also too much and too noisily. Cat repeated that he has yet many complaints from inhabitants of the forest. So he visited Winnie day by day and spoke, spoke, spoke...

Strongest branch in an oldest oak in such a huge forest! Needless to repeat that Big Cat after Winnie’s removal would rent it at once gaining considerable profit, or would privatize all that for cents and will hire for the tourists practically all year long.

Sunk in deep thoughts Winnie paid no attention to the crow which flew by holding a good piece of cheese in her beak. And this crazy crow of course dropped that cheese with weight naturally exactly 1 kg into one of Winnie’s jugs. First Winnie didn’t take any notice of what had happened.

But in the forest in every tree and also around it as well as in any other possible place or in every square meter the life was developing with extremal intensivity. That explains why after a few seconds since the 1 kg piece of cheese landed into one of Winnie’s jugs his best neighbour Magpie came along with her eternal chirping: „Oh, oh I have seen, have thou seen where the crow must have been?“. It lasted several seconds until Winnie understood the whole sense of what had happened. What still wondered him was this everlasting complete Magpie’s knowledge about all possible things which happened around. In our case Winnie wondered how did she happen to know that the weight of the dropped piece of cheese was exactly 1 kg.

He remembered that Magpie always was excellently informed but still every time when he dealt with that knowledge same strange feelings occured and assisted his thoughts.

He ask impudently: „How can you be so sure that the cheese’s weight was exactly 1 kg?“.

- It was scrapped out on it,- said Magpie.

At that moment the Fox was running by. Naturally the Fox was also informed about the weight of cheese and about the whole affair. Fox also boldly claimed that with his balances without weights Winnie can do nothing in the sense of finding out in which vessel now the cheese is without opening all vessels.

But the skilled Budger the science manager of the forest claimed that this is possible, and moreover he went on claiming that this is possible to be established in two weightings.

- But I’m not so sure that Winnie is already so skilled and clever as to find it out in the next future, - frankly added the Fox.

And they made a bet. The Budger claimed Winnie will do that, while the Fox claimed in turn that Winnie is not yet as clever for that.

Soon each bird and animal ten miles round that oak became involved and informed and long discussions aroused about retrospective and perspectives of it. Every bird and animal became a specialist in the area and wondered whether Winnie could indeed *in two weightings using his correct balances without weights really find out in which vessel this 1kg of cheese was laying. We still remember that there were exactly 9 vessels with honey weighing exactly 1, 2, 3, 4, 5, 6, 7, 8, 9 kg each and that the weight of each vessel was being clearly indicated on the bottom of every vessel. Now when the cheese was being dropped in, one of these* *inscriptions wasn't true. Which of them? How to find it out for sure?*

So Winnie was sitting on his branch deep in thoughts regarding what to do, and thousands of invisible eyes were following his slightest wink.

They were waiting and Winnie was racking his brains.

What kind of advice could we make if we were starting consulting him?

How could we now apply our „allmighty idea of simplification"?

Our consultings could start with the serious proposal to Winnie firstly to try to understand at most how many vessels of honey with regularly increasing weights being

1kg, 2 kg, ...,

could Winnie possess, so that after similar drop of cheese in one of these vessels he would be ***able to establish in one weighting in which vessel did that 1 kg piece of cheese land?***

With one jug of honey of weight 1 kg everything is clear as the day and moreover we need to do nothing: there is one vessel and the piece of cheese is in that vessel.

Having two jugs with weights 1 kg and 2 kg after the drop of cheese into one of the vessels we must already do something – no happiness without the deeds.

In that case the things are obvious or almost trivial: you can invent nothing better than putting the 1 kg vessel on one plate, the 2 kg vessel on another plate and to see what will happen.

Now two things can happen: either both sides are even or they are not. In the second case one side is lower (and another higher).

If both sides are even it means that the cheese is in the vessel on which 1 kg was written, and if both sides are not even then the cheese had fallen into 2 kg vessel.

So in the case with two vessels we can easily find where the cheese had felt in making one weighting.

And what if we would take three jugs of 1kg, 2 kg and 3 kg?

Then the matters are not as promising as we would like them to be.

But the question remains and demands the answer.

What to do with 3 vessels of successive weights 1 kg, 2 kg and 3 kg, knowing that there is a 1 kg piece of cheese dropped in one of them and we have only one probe on our disposal?

We start with completely trivial remark that we must lay something on either side of scales. This is true but gives to us not so much.

Imagine we put 1 kg and 2 kg together on one side and 3 kg on another side. Let us say that we are testing whether 1 kg + 2 kg is still 3 kg or shortly whether

$$1 + 2 = 3?$$

If the balance indicates that both sides are even then on both sides there is the same amount of weight. If it is so right now then we will claim that it is impossible. Why? It is clear that both sides won’t be even. They can’t be even because one of these 3 vessels contains 1 kg piece of cheese. This means that on one side there are 4 kg of real weight, while on the other only 3 kg. This is a reason why the sides can’t be even.

But if the sides are not even then one side will be higher. Now there are two possibilities:

(i) side with 1 kg and 2 kg vessels is higher;

(ii) side with 3 kg vessel is higher.

In the case (i) everything is fine because then 3 kg vessel is heavier than the side with 1 kg + 2 kg so that cheese is in the 3 kg vessel.

In the case (ii) we can say only that the cheese is either in the 1 kg or in the 2 kg vessel but in which exactly we can't say for sure – we can only guess.

All this doesn't look very promising.

We ought to regard all cases. In a lucky way there are not much of them because of only 3 vessels involved.

To put all three vessels on one side would be pure monkey's business. If all three vessels are involved then the 3 kg vessel is also being used, and putting other 1 kg + 2 kg on the other side of balances gives us the previously discussed hopeless situation. That situation is bad because there is a case when we have no solution or we are not able to find the vessel with cheese. It is so in the case when

$$1 \text{ kg} + 2 \text{ kg} > 3 \text{ kg}.$$

Further if one of these 1 kg or 2 kg vessels comes to the same side where 3 kg lays then on that side we have already at least 4 kg of weight amount (or 5 if the cheese happens to be on that side) and on the other side there are at most 2 kg (or 3 with possible cheese). So we can again decide nothing.

It remains to discuss the case when we use 2 of 3 vessels in that only weighting. Then we can take 1 kg on one side and 2 kg on another. If the sides are even between plates then everything is clear and fine: 1 kg piece of cheese is contained in the 1 kg vessel. But if the sides are not even then we can't say anything of importance again because then the heavier plate is a plate with 2 kg vessel and it might be with cheese and also might be without. Clearly the 2 kg vessel either with cheese or without it is heavier than the 1 kg vessel without cheese.

The same could be repeated in the case when we lay 2 kg on one side and 3 kg on the other.

In the situation "2 of 3 used in one weighting" only the case 1 kg on one side and 3 kg on another side remains. That case is the worst: we can't say anything definite at all.

So resuming we can say: having vessels with consecutive weights 1 kg, 2 kg, … and only one weighting at our disposal we can manage only the case with two jugs weighing 1 kg and 2 kg. It is impossible to achieve nothing more having 3 or more vessels.

Still we would like to emphasize that nevertheless we've learned a lot. You'll see it in the coming chapter.

BREAKTHROUGH X (XXXIV). 9 JUGS AND 2 WEIGHTINGS: FANTASY OR REALITY?

The main fear, which we may take from the previous chapter, is perhaps the fear to remain with 3 vessels in second weighting. This was exactly the situation when we'd failed in the previous chapter. In this chapter we'll see how the 1st weighting will help the 2nd one and that now we may remain with 3 vessels and won't fail during the 2nd weighting.

Let us begin to check the equality whether former 1 kg + 3 kg + 8 kg vessels still have the same weight as 2 kg + 4 kg + 6 kg vessels? The vessels 5, 7 and 8 remain untouched.

Assume that it occurred that the side with 1, 3 and 8 kg vessels is heavier. It can be so only because one of these 3 jugs contains cheese. From the first sight we might imagine that our situation is again as hopeless as that with 3 consecutive vessels with weights 1, 2 and 3 discussed in previous chapter.

Both situations are indeed similar but not identical. We indeed remain with 3 vessels of 1 kg, 3 kg and 8 kg and only one weighting left.

But one thing now is completely different as it had been in the previous chapter: we have a lot of vessels about which we know that they are without cheese or that their weights are exactly as much as it is indicated on them.

The vessels 2, 4, 5, 6, 7 and 9 kg now are “right” vessels or vessels with no cheese weighing exactly as much as it is written on them.

What are we going to do now in the second weighting?

We are going simply to take the “right” vessel of 5 kg and to use it checking whether sides with 3 + 5 and 8 are even or not. (1 kg vessel was taken away from balances and laid aside).

Using our terminology we are testing the equality 3 + 5 = 8.

Now there are three possibilities:

(i) sides are even;

(ii) side with 3 and 5 is higher;

(iii) side with 8 is higher.

Our decisions will be as follows:

(i) The fact that in the 2nd weighting the sides are even indicates that both vessels of 3 kg and 8 kg are "right" vessels, so cheese must be contained in remaining 1 kg vessel, which was just being taken away and left aside.

(ii) If the side with 3 kg and 5 kg jugs is higher that means that cheese is on other side. But on other side there is 8 kg vessel and the cheese must be in it.

(iii) If the side with 8 kg is higher then the cheese on the other side with vessels 3 and 5. But 5 is vessel without cheese or "right" vessel, so 3 kg vessel must be with it.

How are we are acting when the plate with vessels of 2 kg, 4 kg and 6 kg prevails? This again indicates that cheese is contained in some of these 3 vessels. We take then 4 kg vessel away and lay it aside and in the second weighting we are checking whether

2 kg + 8 kg (that's "right" jug!) =

= 6 kg + 1 kg ("right" jug!) + 3 kg ("right" jug!)

If the sides are even then the 4 kg vessel which we'd just taken away is a vessel with cheese, if the side 2 + 8 is higher then the cheese is in the 6 kg vessel and if side 6 + 1 + 3 is higher then the cheese is in the 2 kg vessel.

And what in the case if checking whether

1 kg + 3 kg + 8 kg = 2 kg + 4 kg + 6 kg

we have that these sides are indeed even?

Then the cheese is either in 5 kg or in 7 kg or in 8 kg vessel (in the vessels which remained untouched during the 1st weighting).

Again we lay aside the 8 kg vessel and take instead of it the “right” 2 kg vessel for testing whether still

5 kg + 2 kg (“right” jug!) = 7 kg.

If now the sides are even then the cheese is in the 8 kg vessel, if the side with 5 kg + 2 kg is higher then cheese is in the 7 kg vessel, and finally if the 7 kg side is higher then the cheese is in the 5 kg vessel.

So we have regarded all possible cases and are extremely fond to state that we are able to determine in two weightings in which vessel the piece of cheese which the crow had been carrying by but droped has landed in.

BREAKTHROUGH XI (XXXV). THE SECOND TEST OF WINNIEE-THE-POOH

There once was a boy of Baghdad
An inquisitive sort of a lad
He said, „Let us see
If a sting has a bee.”
And very soon they find out that it had.

Could you ever imagine that Winnie-the-Pooh became rich and bought the place of residence with the special larder for honey supplies?

On one shelf there were 10 small glasses with honey. First glass is with 100 g of honey, second with 101 g, third with 102 g and so on, the 10^{th} contains 109 g of honey. There is also an unbounded supply of empty glasses; **all glasses are of the same weight.**

You may ask why there are such small glasses if Winnie now is no more poor?

The answer is that Winnie now is indeed no poor person. He is attending also the fitness club, that’s why he is no more consuming honey in kilos.

But by the old good tradition the weight of honey included in it is carefully written upon the glass. And the balance is also the same. Everything remained similar as it had been. Only honey is measured in grams and no more in kilos.

One day when all glasses as well as all doors of larder were open suddenly a swarm of hornets, each of them weighing only 1 g, flew in. The careful homekeeper reported that after the incident not all of hornets flew away. It followed that at least one hornet had fallen down or sunk in some of these opened glasses.

Now when all glasses were being closed Winnie thought whether is it possible without opening these glasses but using balance without weights to find out at least one glass with at least one hornet in it.

The number of weighings now was of no importance for him, **the only principal thing which bothered him was to find out for sure some glass containing at least one hornet.**

We are again his advisors. What should we advice to Winnie?

Firstly of course we could advice him something what we usually do - to reduce drastically the number of glasses because we feel that the proper solution shouldn’t depend on the number of glasses. It shouldn’t be any essential difference between 2 or 10 or even 1000 such consecutive glasses. Everything should be similar – such are our insights.

Let Winnie take only two glasses: one with 100 g of honey in it and another with 101 g. That is only but exactly 1 g more.

What to do?

There is no big choice. There is practically no other choice than to put the 100 g glass on one side and the 101 g glass on another side and to see what happens.

As usual there are three possible outcomes of the weighting:

(i) Both sides are even. But one side is the side with originally only 100 g of honey in it. If 100 g isn't with hornet then it would be impossible that the sides with 100 g and 101 g are even. So only hornets in that 100 g glass may ensure the evenness of balance.

So if the sides are even then we can guarantee that in 100 g glass now there is at least one hornet. Nothing more couldn't be told – neither concerning number of hornets nor about their presence in another 101 g glass.

The only essential thing that could be added is the following: If there are some hornets in the 101 g glass then in the 100 g glass there would be exactly one hornet more.

(ii) The side with 100 glass is higher than the other side with 101 g glass.

In that case the same essential conlusion concerning the presence of some hornets in the 100 g glass could be made again. Even slightly more could be stated. Now we can guarantee that there are at least two hornets in the 100 g glass. Or, generally speaking, that in the 100 g glass there are at least 2 hornets more than in the 101 g glass.

(iii) The side with 101 g glass is higher than the side with 100 g glass.

What's now? There is the „usual weight hierarchy" obeyed just as it was.

In the case we are leaded by some consideration of philososophical nature:

Because it was assumed that at least one hornet felt into some glass we can guarantee now that in the 101 g glass there is at least one hornet. Otherwise we would have the contradiction: if in the 101 g glass there are no hornets then there would be some or at least one hornet in the 100 g glass so then the side with 100 g wouldn't be easier than that with 101 g.

Now returning to the shelf with 10 glasses with 100, 101, 102, 103, 104, 105, 106, 107, 108 and 109 g of honey in them we see that we understand already perfectly what should we advice to Winnie.

He ought firstly put first two neighbouring glasses, that with 100 g on one side and that with 101 g on the other side. If both sides are even – that is, if the former "weight hierarchy" is no more valid, then Winnie can drop weighting and announce about at least 1 hornet contained in that 100 g honey glass.

If both sides are not even then Winnie should go on comparing next neighbouring glasses, or these of 101 g and 102 g. Again if the former "weight hierarchy" is no more valid then there is at least one hornet in the 101 g glass. If it is, then we compare the following neighbouring glasses, and so on.

In any step, if the former "weight hierarchy" of neighbouring glasses is no more valid, then there is at least one hornet in that originally easier glass.

The final step is to decide what to do if in all cases weighting neighbouring glasses shows that the former "weight hierarchy" holds. Then...

WHAT'S THEN?

Then it is exactly so as it was in the case with only two glasses of 100 g and 101 g on the shelf.

Then it can again be stated that in the last glass, or in this case in the 109 g glass, there is least one hornet.

Note that we are in no way able to establish the least number of the glass which contains a hornet.

BREAKTHROUGH XII (XXXVI). THE THIRD ALREADY METAPHYSICAL TEST OF WINNIE

After intensive thought Winnie was sleeping as a new-born baby. Babies are believed to sleep without dreams. Winnie dreamed sometimes. So once in some thrilling dream Winnie had seen how his shelves became to be infinitely long (you know, in dreams everything is possible). In dream he had clearly seen the infinitely long shelf with infinitely many glasses of honey: 1^{st} glass was with 1 g of honey, 2^{nd} – with 2 g of honey an so on,..., the 2007^{st} was with 2007 g of honey, and so on – it was no end for honey and glasses.

Again he had seen in the very same dream the unbounded quantity (finitely or infinitely many – afterwards he'd found out that it makes no difference) of hornets all of the same weight again equal to 1 g which was flying along that shelf, and at least one of them felt in some glass with honey.

Winnie woke up deeply terrified with the thought that in such situation even after infinitely many weightings with balance he cannot guarantee for the glass with hornet in it.

Again if we would lay down any pair of glasses with neighbouring weights containing honey then even assuming that he is able to do infinitely many weightings with these neighbouring glasses nevertheless it might happen that Winnie couldn’t guarantee for anything.

Well, again if in some weighting the former „weight hierarchy“ would be no more valid then Winnie of course would be able to show one glass with honey and hornet. Say, if he laid on one side of balance the glass with 2007 g of honey and on the other side – 2008-gram honey glass and it appeared that the 2007-gram glass isn’t esier than that of 2008 then Winnie could say for sure that there is a hornet in the 2007-gram glass.

But if in no case the „weight hierarchy” is being “violated” then he could say nothing – only guess. But guess doesn’t mean find out.

For example, imagine that in each glass one hornet felt in. Then the “hierarchy of weights” isn’t „violated”. Because there is no more such a notion as “the heaviest glass” we cannot show some glass and say: „*Upon my word of honor, there is a hornet in that glass”*.

This is a good example that **“infinitely many can be too much”**.

In other words the situation which we were able to solve in finite case we are no more able to solve in infinite case. The task remains the same. But in finite case we can manage the situation and in infinite case we can’t, at least we can’t do it in the same way.

*Of course, these considerations are not any kind of **proof** that Winnie's task is unsolvable in the infinite case. The proof can be given, e.g., by contradiction, comparing two situations:*

a) one hornet in each glass,

b) no hornets at all.

We assume that the reader isn't angry hearing our speaches about the infinitely many Winnie's glasses.

The infinity is around us, it looks to us from variuos places or from each corner. For example, no one has slightest doubts that there are infinitely many integers (practically, given any N we will surely find larger integers than N)!

We also have no doubts that in any intervall there are infinitely many real numbers.

This reminds the public joke about the man of property and wealth who mentioned once that the really rich man is that which is not able to count up all his money.

That is also something going towards the infinity. In our imagination the infinity is something extremely large, something possessing no limits, something almost non-understandable and mysterious.

To all what was being actually told, in the following Breakthrough we would add also some slightly mystically looking problem. That problem by the first glimpse describes the situation where no solution seems to be possible or at least it is not so easy even to guess and feel from which side the solution would arrive.

BREAKTHROUGH XIII (XXXVII). HOW TO FIND THE PERSON WHOSE NAME NOBODY KNOWS

Once it happened that the friends of Sherlock Holmes, that most famous detective of all nations and times and also the inventor of deductive and many other scientific and artificial methods in pure and applied sciences, organized his meeting with another famous detective Poirot. Their meeting should better be called seminar or workshop if we use modern slang and terminology. It should be also noticed that all these things where organized in those days when no one in England knew the name of that famous French speaking detective star.

This meeting taking place in London naturally was provided in the famous Baker Street, in the new build Palace of Arts and Logics. It should be repeated that no person in England knew the name of Poirot. It should be also pointed out that Mr Poirot arrived in Baker Street not alone but in assistance of 99 assistants. But even these assistants also didn't know the name of Mr Poirot, which in turn knew perfectly well the name of each of his assistants. Concerning the names of other assistants it should be added that some of them knew some names other assistants and some not. When being asked for the names of other, all assistants involved were always speaking truth.

It means: if you were speaking with any of these assistants and, pointing out to some other assistant, were asking whether he knows the name of that other assistant you just pointed out, then your question would always be truly answered with “Yes” or “No”.

For the sake of curiosity and completeness it should be also added that Mr Poirot and all his assistants were identically dressed and also identically made up so that they all looked really like the copy of one and the same person.

On the opening ceremony Mr Holmes in excellent French, in aristocratic and in the same time unsupposedly cordial way greeted all his colleges and friends. Afterwards he also confessed that he also doesn’t know the name of Mr Poirot. He declared also that he could try to find out which of his honored identically looking guests is indeed Mr Poirot.

For that he asked only the permission to approach any of his guests who all were standing together or *in corpore* in the guest hall and to repeat if necessary the only question we already mentioned and discussed.

Mr Holmes was asking for permission to approach any of them and, pointing out to any person, ask the question: “Do you know his name?” We would like to remind that possible answers could be only “Yes” or “No”.

No other answers were allowed. Mr Holmes ensured that there will be no need to repeat that question more than 100 times. Really we felt some doubts whether these “yes” or “no” could ever lead to the aim which was the indication which is Poirot from that centurion or hundred identically decorated gentlemen.

We would like to mention once more that hearing all this we felt some slight doubts about the possibility to find out the person the name of which nobody knows, repeating the same question.

Otherwise we understand also that Mr Holmes didn't promise to find out that unknown name, but only the person carrying that unknown name. There is a slight difference between these two matters and the second task might be a bit easier.

Nevertheless, frankly speaking, we all were interested to see how Mr Holmes was going to fulfil what he'd promised to.

All who attended at that moment remained silent just as they were – it's no wonder by such speciality, which is always also the way of life and mirror of attitude and behavior.

So in our mind's eye we see Mr Holmes approaching to some person – let it be Mr A – and pointing out another person – let it be Mr B – and asking Mr A whether he knows the name of Mr B.

What conclusion may be drawn from the Answer YES?

From the Positive Answer YES we may draw the conclusion that the name of Mr B is known to Mr A so that Mr A is disposing some valuable information. So Mr A could be even Mr Poirot itself but not necessarily – he could be also one of these better informed gentlemen. Concerning Mr B it could be stated for sure that Mr B couldn't be Mr Poirot because in that case Mr A wouldn't know his name.

Shortly speaking in the case of Answer YES Mr B is not Mr Poirot or MONSIEUR POIROT.

Ask Mr B in such a case to follow the housemaster to another room with food supplies and refreshments.

And what conclusion may be drawn from the Answer NO?

From the Negative Answer NO we may draw the conclusion that the name of Mr B is unknown to Mr A. So now Mr A doesn't belong to these AT LEAST A BIT INFORMED PERSONS NOT TO SPEAKING THAT HE MIGHT BE Mr POIROT.

Shortly speaking in the case of Answer NO Mr A is not Mr Poirot or MONSIEUR POIROT.

So now ask Mr A to follow the housemaster to the room with food supplies and refreshments as well.

We understood now exactly the main principle how Mr Holmes is acting and performing. In the first step in either case he eliminates already one person. In the case of Answer YES he eliminates that person he actually pointed out after approaching and in the case of Answer NO he eliminates the person, which he just approached and asked.

On the second step an arbitrary pair already can play the role of a pair consisting of Mr A and Mr B with the same question and the same way of elimination of one person from that pair. In either case one of them has to follow for relax.

Now for the fundamental reader the last question might arise:

What will happen at the very end? Mister Holmes is continuing eliminating persons till only two persons are being left in the room.

What will be the final accord of that story? What will happen after the question will be repeated 98 times?

It's clear. The same question of course. And what will be Mr Holmes decision about the possible answers?

After these 98 identical questions the same, already 99th, version of that question to the only pair of guests left in that room will follow.

This question is

“Do you know his name?” asked Mr. Holmes attending Mr A and indicating to Mr B.

In the case of Answer YES Mr B could in no way be Monsieur POIROT.

Then another person of these 2 guests left or Mr A must be Monsieur Poirot because Monsieur Poirot is still present in that room. He must be one of these remaining 2 persons. There is no other choice.

If not Mr B, then Mr A.

In the case of Answer NO Mr A could in no way be Monsieur POIROT.

Then again another person of these 2 guests left or Mr B must be Monsieur Poirot because Monsieur Poirot is still present in that room. There is again no other choice.

If not Mr A, then Mr B.

We did it. Or better to say, Mr. Holmes did. We all did.

We understood once and for all times how does it run.

We found the person the name of which was unknown.

We found it using the simple but effective method of elimination.

Simple conclusion after successful process of elimination.

Again if the number of friends were infinite then even Mr Holmes applying that method of elimination wouldn't achieve anything.

Because if you have infinitely many persons then, eliminating person one by one, you will never achieve the situation which we just enjoyed with the only pair of persons left after a finite number of such steps of elimination.

There will be infinitely many guests left after any (finite) number of steps.

Always infinitely many. Never one, or few or several.

We dare express it by saying that

INFINITY MINUS ONE IS ALWAYS INFINITY.

To understand what was being performed is one of the finest feelings.

BREAKTHROUGH XIV (XXXVIII). WHAT IS POSSIBLE TO FIND OUT AND WHAT IS NOT?

The beaver had counted with scrupulous care,
Attending to every word:
But it fairly lost heart, and outgrabe in despair
When the third repetition occurred.

Very often being in good mood or simply optimistically disposed we naturally tend to think and believe that we are able to establish or find out practically everything. We need of course some essential information.

Otherwise we would repeat with the Romans:

Ex nihilio nihil

The translation might be:

Nothing comes from nothing.

There are several reasons which are responsible for that.

Firstly we may know not sufficiently much. It may happen and often happens that they have given us not enough information. It is worth mentioning that it might happen and also happens that we are given too much information. That is, we could and would to find out the truth also in case if we wouldn't know that much as we actually know.

In the first case when we're insufficiently informed and still trying to be at least a bit funny we could cite the famous problem from the book "The adventures of the brave soldier Šveik" due to the worldwide known Czech writer Jaroslaw Hašek.

In the house there are 5 floors with 5 windows in each floor. In what year the Grandma of the house master was born?

It also happens rather often that, when we face the problem, we know too much – or that we have redundant data condition.

Then of course we are to choose and can also state explicitely what we really do need to apply for finding out what is the truth in the given situation. It could even be said:

To know too few isn't good at all but to know too much often is also not convenient. It can be bothering. We can be forced to reduce. It takes time.

At this place the author remembers some old joke from the socialist time which he have seen in the Polish humour magazine Szpilki.

The client to the waiter after his bill was presented:

- I have no money. But I have time. And time is money.

There are also conditions, which do not allow us to reproduce the full retrospective of what had happened but are sufficient for reconstruction of some aspects or some details only.

We will consider some example, which we have seen in a Sankt-Petersburg book of mathematical or panhuman wisdom.

We will again slightly adobe or dramatize the condition carefully trying to keep the intrigue of the whole adventure.

The modern generation is considerably less familiar with the outstanding warriors from the French history, namely with ATOS, PORTOS and ARAMIS and with their colleague D'ARTAGNAN which you perhaps couldn't call the chief of that company but who was something of the kind.

The famous French writer Dumas wrote plenty of books describing their adventures. You can't guarantee that all what was described really had happened because it lookes too good to be true. Some kind of Harry Potter - many are reading and enjoying but very few do take care whether it indeed might happen. The fantasy of writer eliminates such a question up to some time.

Concerning the deeds of musketeers the historians claim that the historical context was set perhaps similarly but quite differently, but nobody's taking care for that because it is so nice described.

And, concerning objectivity, it could be added that if your version is the nicest among all versions which remain when the event is already forgotten, then your nicest story is believed to be also the truest one.

Si non a vero a ben travato.

Even if untrue, well done.

What's well done that's nice, and what's nice that's very convincible.

BREAKTHROUGH XV (XXXIX).
LAWN TENNIS OR THE BRIGHT MUSKETEER D'ARTAGNAN

He walked with hyenas returning their stare
With an impudent wag of his head
And he once went for walk paw by paw with the bear
Just to keep up its spirit, he said

This is a modern problem decorated with names, which once were known as well as now the name of Harry Potter is. These names of musketeers were well known to each who was fond of reading the belles-lettres, and the number of persons who were fond of reading in those days could be comparable only with the number of these who nowadays are spending days and nights with computer and so become familiar with informatics.

All we wanted to express is that these who nowadays are good in informatics, in those days were intensive book readers.

Once these three musketeers were training on the tennis-court – don't forget that they all were warriors of the King Louie XIII, the counterpart of Cardinal Richelieu.

Meanwhile they all were involved in the tennis game. The game was arranged in the following manner: two were playing one set and the third was their judge. After the game was over the loser was going to judge and the judge was going to play with the winner, and so that whole affair run on.

After that long play was over they stated that Atos had played 15 sets or 15 times, Portos – 10 and Aramis – even 17. D'Artagnan, who just arrived seeing that whole statistics consisting from 3 numbers, declared his ability to determine losers of many sets.

– *But how on earth can one determine the loser of so many sets having these 3 numbers only?*

– *Many facts grow up from these 3 numbers.*

– *For instance which exactly?*

– *For example I could tell you who'd lost the second set you've played.*

– *May you be able to tell also who won the second set?*

– *But I beg your pardon; I'm not speaking about who **won** the second set but*

– *I'm speaking now about who'd **lost** it.*

– *Does it make any difference?*

– *Yes, it does. I repeat that I'm not claiming to be able to determine everything in France having only these 3 numbers, but I insistently repeat with all my might that I can say for sure who lost the second set.*

– *Can you say also who'd lost the third set?*

– *No, I'm not speaking about it.*

– *What else could you say having these 10, 15 and 17?*

– *Many things. For instance I can tell who **lost** the 16th set.*

– *And may be you can tell us also who **won** this 16th set?*

– *Again I am not speaking about who **won** it but about who **lost** it.*

– *So what, you who are such an excellent fighter and perhaps the first amongst king's warriors suddenly become a specialist on losing?*

– *I do what I can.*

– *Please explain us how are you doing it.*

– *Let us sit down on the bank and, please, listen for a while.*

Dear reader, we understand that not all of you and me would be very fond of such an art of popularization of so simple problem. Not everyone likes such adoptions. But still it's always worth trying.

On the other side if I'm willing to deal with problem I'm expected to know what's the problem I've chanced to meet is about and be also able to keep in mind all essential circumstances of the deal.

The adoption can also be slightly irritating and not convincing when I'm running around with the serious face repeating that one jump of kangaroo can be 100 km long or that a horse can overleap the fence that is 15 meters high. Or it's a poor fantasy and I am the fan of it.

Still the author would never give up his opinion and will go on repeating that any at least a bit successful adoption or more vivid presentation of the possible task makes the sense of the whole happening more attractive to the possible future solver. It simply and naturally makes the possible future solver cleverer and keeps up his spirit, and that together with all other useful things is the most valuable thing what the education may bring and propose.

To make the solver more clever than he'd been before and to keep up his spirit – this is the most honorable duty of every human art, especially also of mathematics.

Sometimes we are too shy to say it ***expressis verbis*** or express it in words.

Let us go back to explanations of D'Artagnan. In a wondering way he was speaking about simplest things. Let's listen to:

He firstly advised to add up these 3 numbers – 10, 15 and 17. Summing up we've got

$$10 + 15 + 17 = 42.$$

What could we extract from this prosaic banal sum 42? It gives us the common number of all persons involved in all provided sets.

Now we ought to emphasize strongly that one and the same person from any 3 of them will be counted several or many times.

Two players make a set. Two players mean 1 set. So 42 players mean

$$42 : 2 = 21$$

set. So from that banal sum we extract that there was 21 set to play.

Take a glimpse at the 3 given numbers again.

Atos had played 15 sets or times, Portos – 10 and Aramis even 17. There was 21 set as was told and repeated. Who was the most successful player? Again the most natural thing is to assume that the person who played most is the strongest. He was being eliminated not so often. There remains practically no doubt that the person who played most is the strongest one among them. So according to numbers Aramis seems to be the strongest, Atos - almost as strong as Aramis and finally Portos is the weakest. So perhaps he as the weakest is that one who lost the second set. Why must it necessarily be him? What's the reason? Everyone, even the strongest player, can lose. **Only God can't lose**.

Indeed, will it really appear that is was Portos?

Look again at the number of games Portos played. Portos played 10 times or sets. There was 21 set. It is clear that each musketeer was playing at least every second time. That means that in two neighbouring sets each of them must have played at least once. But 10 times participation in 21 set leaves for Portos the only possibility to implement such game configuration. This possibility is the necessity to play the second set, then the fourth, the sixth, the eight, the tenth, the twelfth, the fourteenth, the sixteenth, the eighteenth and the twentieth set.

And unfortunately the only possibility to make such a game configuration or to play 10 times in 21 set means also that you have lost all these sets.

Otherwise you have had participated in some 2 neighbouring sets. But you did not. You lost them all.

We remember now that D'Artagnan was always speaking about losers and not about winners. He spoke also about who'd lost the 16^{th} set. He avoided any talk about possible winners.

Nothing more could be told with exception that in other 21 – 10 = 11 sets Atos was playing vs Aramis and that Aramis should have won more games.

Question to the reader:

Can you deduce from these three numbers 10, 15 and 17 how many times did Atos win and how many times Aramis did?

BREAKTHROUGH XVI (XL). BEAUTY AND ELEGANCY OF SIMPLE THINGS

There was an Old Man of Apulia,
Whose conduct was very peculiar;
He fed twenty sons with nothing but buns
That whimsical man of Apulia.

The beauty is very attractive; so attractive that it is even capturing.

You can't speak more powerfully about that perhaps than Lord Byron did. Let's listen to some classical verse. But even then much remains untold. And always will.

She walks in beauty like the night
Of cloudless climes and starry skies
And all that's best of dark and bright
Meets in her aspects and her eyes.

Psychologically and humanly beauty is very powerful. Beauty is something which seems not exhausting. You are consuming and enjoying it - still an impression that much more remains untouched and not used. Similar as it is with infinity.

Infinity minus several is infinity again.

Beauty which appears on the common place where you even never could suspect it would appear is psychological point of view even more powerful and striking. Similar as I've red in an English textbook of Ekerslay:

When I was walking down the street Sir Winston Churchill went by….

Something totally unexpected may appear on a common place which you know so well, which you've seen so many times and where you'd never expected something what could change your views.

Otherwise the human being is also an old gold seeker, compare **Matthew 7:8**, where it is clearly written: he **who seeks** that **finds,** and he who knocks that has the door opened to him…. It could be added that you can't expect it be opened for you promptly or at once but one day or another it surely would.

So paraphrasing: he **who expects** that… .

It is natural that persuing for truth the human being has also to deal with some obstacles of logical as well as psychological, technological and ideological and of any other nature. Otherwise it would be many times found and forgotten.

This is understandable because if I've solved even a simple or even a trivial problem nevertheless I have succeeded in doing it and for me as a solver it is pleasant and I will always be conscious of the fact that I am able to present the answer when being asked to.

And may this, even if only in a slightest degree, widen and deepen unmarkable boundaries of wisdom, mine and yours – we knew, we did, what a wonder. Let's continue repeating this. Would you like to present us another problem?

And the things look quite different when I'm being asked for something what appears not so simple, when I'm asked for something what I do not quite understand, what I've never heard, when I'm being asked for something that, when trying to give an answer, makes me break my head, mobilize my whole experience and will, spend enough of my time…, my time which is so valuable and what I never have enough.

Even then when I'm able to find what I've being asked for.

Then I must repeat to myself: don't lose your temper, take your time and try again.

Easy said, not so easy done.

Let start from that what is possible

It would be not so easy to find the person, who wouldn't be able to master the following task in an hour:

The wood which is exactly 100 metres long was cut into 30 pieces each of which being either 3 or 4 metres long.

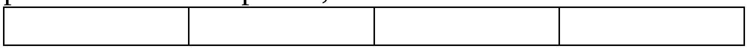
How many times must you cut these pieces into smaller ones so that all pieces would be exactly 1 metre long?

We are sure that the first but innocent consideration will always be the same: making all pieces to be 1 metre long you must do 2 cuts with every 3 metre piece and 3 cuts with the 4 metres long one.

If we knew the number of 3 metres long pieces as well as that of 4 metres long ones, then multiplying the number of 3 metres pieces by 2 and adding it up with the number of the number of 4 metres pieces multiplied by 3 we would get the whole number of necessary cuts when we're sawing "bigger pieces into smaller ones".

The reader with the normal fantasy and even without sawing-cutting skills could easily imagine that whole process of dividing. Others may visualize and illustrate that process of dividing with simplest picture like this:

One picture stands for 3 metres long stumps indicating with bars the future cut places when dividing piece into 1 metre pieces, and another is for



4 metres stumps also indicating with bars the future cut places.

Further on the usual process of data densification or translation of the whole situation into equation industry follows.

We are to repeat some of these sacramental slightly boring school phrases saying: let us denote the number of 3 metres pieces by $\boldsymbol{X}$ and that of 4 metres by $\boldsymbol{Y}$.

Then first of all – because in the beginning it was clearly indicated that there were exactly 30 pieces – transferring the data we write

$$\boldsymbol{X} + \boldsymbol{Y} = 30$$

On the other side all these 30 pieces we've got from the 100 meters long wood so remembering that originally there were no other stumps but only these of 3 or 4 metres long we get

$$3\boldsymbol{X} + 4\boldsymbol{Y} = 100.$$

Even such simple system of linear equations might be solved in a slightly different manner than it is usually done.

Noting that the number $3 \cdot 30 = 90$ lays closer to 100 than another number $4 \cdot 30 = 120$ we start our considerations saying: assume that all these pieces were 3 metres long. Then, strictly speaking, this situation is impossible because if it were so then the total length of all these 30 stumps would be $3 \cdot 30 = 90$ instead of 100.

We intend now to start the barter process exchanging 3 metre pieces into 4 metre pieces. We are going to realize it without any hurry exchanging them one by one.

By that slowly, solid exchange the total number of pieces after each "atomic" change remains as it had been before.

In the same time total length after each change is increasing – in each step we gain 1 metre of length. Supposed to gain $100 - 90 = 10$ metres of length we must exchange exactly 10 stumps of length 3 into 10 stumps of length 4.

Summing up we see that there were 20 pieces of length 3 and 10 pieces of length 4 so that the number of necessary cuts when dividing them all into pieces of length 1 is

$$2 \cdot 20 + 3 \cdot 30 = 70.$$

After reading this the reader may state and repeat that all this was the dull problem allowing most standard solution, and our answer would be that the reader is right.

But replying that the reader might be right, in the same time now we would like you to take a look to another possibility for doing that. We are sure and convinced that at this moment even the previous standard solution would appear for you of some value and importance because it is helpful now for comparing what is standard and what is nice.

The first move now will be a totally unexpected one: instead of cutting "bigger pieces into smaller ones" we start with the process of an opposite nature or "putting smaller pieces together" or, more generally, we are "gluing" instead of "cutting" now.

More precisely, we are putting together all these 30 pieces into one former wooden superstump or log. Doing that we imagine that for "gluing" these 30 pieces into one "super piece" or the original stump we need, of course, not 30 but just 30 – 1 = 29 "gluings".

After that our super piece again is 100 meters long just as it had been. Now we are going to divide it into 100 pieces of length 1. For that we need to do not exactly 100 but just 100 - 1 or 99 cuts. So 29 cuts will be done in the "old" gluing places and the remaining

$$99 - 29 = 70$$

will be indeed the "new" ones.

This problem with the nice final remark is being taken from one of Moscow Olympiad books.

In that place the author would like to add the following comment of heuristic nature. A really enjoying circumstance, which somehow strikes you in the last solution, is that we can do it even without knowledge of the lengths of stumps. The only things that you must know about the stumps seem to be

(A) the total number of pieces (in the considered case there were 30 of them);

(B) the lengths of stumps are expressed in entire metres – or from practical point of view – you can divide each stump into some 1 metre long pieces.

Nothing more is required; nothing more is needed, no other obligatory details ought to be presented.

So we think that the reader understood how using one line or two words, or three signs, the answer to the following "general" problem could be written:

The N metres long wood is divided into *n* stumps of integer lengths

$n_1, n_2, \ldots, n_k$.

How many additional cuts ought to be made to make all stumps to be 1 metre long?

It would be not honest or simply common, or it would be cruel to take away from the reader the pleasure of finding out the correct answer, which is as short as possible.

BREAKTHROUGH XVII (XLI). CUTTINGS ARE FOLLOWED BY BROKINGS

The origin of solving ideas, even these of simplest nature – things, which lay at hand, sometimes are so difficult to take – is usually fascinating, and there are always some reserves for improving left. We feel a rather strong temptation even to formulate it as some principle or repeat the known tourists' law:

Don't be so sure that your way is the shortest one!

Philosophically speaking, everything has its place and price. Development of ideas always takes time. Application of these ideas, when solving and considering, also isn't performed in three seconds.

You may also cite the answer of Pythagoras when he was asked by the emperor to show him the easiest way of learning geometry. We remind the reader that the answer was: "O king, there is no king's road into geometry!"

The courage of Pythagoras at that moment by that answer is outstanding. This is psychologically convincing, sounds nice and is true in general.

Realistic person would add also that this is not the way to speak with kings and other noble persons. Not in vain there is an aphorism of French origin expressed in words "Noblesse oblige".

English translation of that might be "nobility obliges".

Anyway, you really cannot become a master in geometry within few days.

Everything takes time.

We again intend to proceed starting to tell the reader two stories.

First story is about some problem, which is as simple as it might possibly be.

Second story is about another problem, which is perhaps not as simple as it might be. We believe to be able to do it in such a manner that after 10 minutes you ought to be convinced that it is really simple and that the reader will be the master of the proposed problem.

The melody of the problem is based on another problem of Sankt-Petersburg origin ([6], p.9)

John Brown the ever brave soldier has 100 sticks each of which is either 3 metres or 1 metre long. That resolute soldier feels a strong wish, using them all, to mark completely the boundary of some rectangle. If he is able to do it his Granddad will buy for him the parcel of exactly the same form he'd already marked in the downtown of the Dreamwithmath City.

The only condition raised by his Granddad is the possibility to break at most one of these sticks

How could the legendary John prove himself to be bright also in that situation? We remind that his task is to mark completely, using all his 100 sticks of either 1 or 3 meters length, the boundary of some parcel of rectangular form and broking by that at most one stick and using, of course, all of them.

Imagine that John Brown asked us to be his advisors. We could even imagine John Brown also addressing us with words: I know that you are bright and smart. But I do not know whether you are as bright as to give me an effective advise without knowing how many of my 100 sticks are exactly 1 metre and how many of them are exactly 3 metres long?

So he would like to be advised effectively independently of how many sticks of each length he happens to have.

In that case it is not so difficult to fulfil that advisor's duty in an honourable way. Our advice could be formulated in the form of some instruction with short commentaries and be expressed in the following form:

1. Sort all sticks in 2 heaps according to their length.

You will immediately find 2 sticks in some of these heaps.

2. Chose some 2 sticks of equal length to be the opposite sides of rectangle.

Don't care that these sides of your rectangle might appear rather short.

3. Order all remaining sticks independently of their length into one straight line.

Your task is entering already the final phase.

4. Find the middle of that straight line you've just got.

Look whether this middle is exactly between some neighbouring sticks or not. That will prove to be of some importance in the next future.

5. Divide the line you got into 2 parts of equal length breaking, if necessary, some stick to which the middle point might happen to belong.

We are resolutely finishing.

6. Make these two equal parts you've got be another pair of the opposite sides of your rectangle.

Please note that you are done using all your sticks and breaking no more than one stick, and that all boundary is being marked completely.

And now we would like to ask the reader to write down the instruction for your friend, which is in the similar situation possessing 100 sticks of unknown but integer lengths measured in metres. It is also given that the total length of all sticks in common doesn't exceed 5 km or 5000 metres. The task is identical – using all of them, mark completely the boundary of some rectangular parcel, ***still possessing the right to break the only stick if necessary.***

After these settings of boundaries we feel us ready to fulfill some more complicated task formulated as Problem 22 **(in [6], p. 13):**

How, possessing 8 and 9 centimetres long sticks only with total length being 18 metres, to make the whole world including ourselves to believe that it is possible now without breaking any of these sticks to mark out completely the boundary of the regular octagon, assuming that the sticks of both lenghts are indeed present?

What's now? How to start? Is it worth doing? These are so to say eternal questions, which we raise up every day.

Our answer is the resolute YES! Try and gain some experience. Otherwise you will complain yourself by saying: “Oh, dear! I was also able to realize it! Why didn’t I?”

The construction of the regular octagon demands the construction of a polygon with eight equal sides – *is it always possible without breaking of any stick?* We didn’t forget and are still aware that our sticks are either 8 or 9 centimetres long and sticks of both lengths are present.

Firstly we could assume that the construction is possible to compute the length of the side of that octagon. Eight sides with the total length 18 metres or 1800 centimetres means that the length of the side is

$$1800 : 8 = 225 \text{ (cm)}.$$

Now we will write some standard equality. Denoting the number of sticks that are 8 cm long by $\boldsymbol{X}$ and the number of these of 9 cm - by $\boldsymbol{Y}$ we would have that the total length of all sticks which are 8 cm long is $8\boldsymbol{X}$ and the total length of all sticks that are 9 cm long is $9\boldsymbol{Y}$, giving the obvious equation expressing their total length being 18 m, or in centimetres:

$$8\boldsymbol{X} + 9\boldsymbol{Y} = 1800.$$

What else does this equality express? This equality expresses several simple but in our case highly useful things. For example, from that equation we can conclude that $\boldsymbol{X}$ is divisible by 9. This is indeed the case because 1800 and $9\boldsymbol{Y}$ are clearly divisible by 9. But then

$$8\boldsymbol{X} = 1800 - 9\boldsymbol{Y} = 9(200 - \boldsymbol{Y})$$

is also divisible by 9. But if $8\boldsymbol{X}$ is divisible by 9 so $\boldsymbol{X}$ must also be divisible by 9. In the similar way, repeating almost word by word for $\boldsymbol{Y}$ what we’ve just said for $\boldsymbol{X}$, we would get that $\boldsymbol{Y}$ is divisible by 8.

Now we can collect all 9 cm sticks into small bundles with 8 sticks in each. All bundles will be complete, no one stick of 9 cm length will lay aside. Similarly all 8 cm long sticks may be collected into bundles with 9 sticks in each. Again no stick of 8 cm length will be left.

These are the profits or fruits of the divisibity we've just discussed.

How many bundles do we get by that? It is easy to state because in any bundle the total length of sticks in it is the same:

$$8 \cdot 9 = 9 \cdot 8 = 72.$$

Further on we can find also that the total number of these bundles is

$$1800 : 72 = 900 : 36 = 150 : 6 = 25 \text{ (bundles)}.$$

Now we make one swift or almost unnoticeable move – namely we take one bundle with 8 sticks each of which is 9 cm long, and distribute these 8 sticks between 8 sides of the octagon we are constructing.

Now for each side of 225 cm length we have attributed already 9 cm, so in each of these 8 sides there are $225 - 9 = 216$ cm of "free" length. And there are still $25 - 1 = 24$ bundles that we still didn't touch, with total length of sticks 72 cm in each bundle. So distributing the remaining 24 bundles into these 8 sides, taking $24 : 8 = 3$ bundles to each side, we will use all remaining "free" length of 216 cm because $72 \cdot 3 = 216$.

So we are done because we used all sticks and didn't break any of them, and marked completely the boundary of the regular octagon.

BREAKTHROUGH XVIII (XLII). PLAYING WITH STONES AT THE BOTTOM OF ACROPOLIS

The origin of the following problem is due to one of the most famous persons of Russian non-standard education of young generation, I. S. Rubanow**.**

At the bottom of the most famous Acropolis in Athens Sisyphus is expected to select the smallest possible number of stones of five different weights (he might take also some, not just one stone of the same weight if he wishes) in order to be able to play with Hephaistos the following game. Remembering that at the bottom of Acropolis there are enough stones of each possible weight we might assume that Sisyphus is going to gather only such stones whose weights are expressed in integers.

That game which Sisyphus is expected to play with Hephaistos is the following one. Hephaistos, moving first, may touch any 2 stones from these selected by Sisyphus. After Hephaistos has touched any 2 stones, Sisyphus must be able to touch another 2 stones from his collection so that the common weight of the 2 stones touched actually by him must be the same as the common weight of the 2 stones touched by Hephaistos immediately before.

Being not able to answer the touching of Hephaistos, Sisyphus loses. And if he loses, he must return to his ***eternal rock****, which made him so* ***unluckily famous****, and his vacation is over.*

The gods had condemned Sisyphus for ceaselessly rolling a rock to the top of a mountain, whence the stone would fall back of its own weight. They had thought with some reason that there is no more dreadful punishment than futile and hopeless labor.

In order to be able to solve the problem with the touching game with Hephaistos Sisyphus was given a vacation. If he will succed, his vacation could be prolonged. If not, he would immediately return to his eternal rock and will go on ceaselessly rolling it.

Sisyphus had learned much when ceaselessly rolling the rock. The first thing was that any suitable movement or any improvement in the serious situation is of considerable importance and highest value.

He remembered pretty well that first of all he was condemned or forced to deal with the stones of ***five different*** *weights.*

So firstly he regarded the situation when there are enough stones and all of them are of the same weight.

Sisyphus (together with us) quickly came to conclusion that in such case of all stones being of the same weight the same weight there is possess 4 stones. Then the play will be quickly stopped because of everything is as clear as the day.

If Hephaistos touches some 2 of these 4 selected stones then Sisyphus touches another 2. We remind that Sisyphus cannot touch the stones actually touched by Hephaistos and that the common weight of touched stones in both touchings must coincide.

So the case with the stones of one weight is exhausted.

Let’s touch the case with the stones of 2 different weights.

At least how many stones must select Sisyphus then?

The collection of 8 stones with 4 stones in each weight category would certainly do.

Indeed, *if Hephaistos touches two stones* ***of the same weight*** *then Sisyphus touches another two* ***of the same weight***. Now we are essentially using the circumstance that there are 4 stones in any of 2 weight categories.

And *if Hephaistos touches two stones* ***of different weights*** *then Sisyphus touches* ***another two stones of exactly the same*** *weights as these touched immediately before by Hephaistos.*

By the way, from this consideration we see that such a collection of $2 \cdot 4 = 8$ stones is not only sufficient but also necessary to possess.

Namely, if Hepahaistos touches 2 easiest or 2 heaviest stones of the same weight then the only possible answer for Sisyphus is to touch another 2 stones of that weight.

So it is also necessary to have 4 stones of each weight in the case of 2 different weights.

This consideration convinces us also that in the case of 5 different weights the collection of $5 \cdot 4 = 20$ would also do. The number 5 in that consideration may be replaced by any other number, e.g. by number 2007. Then there would be 2007 weight categories, and $2007 \cdot 4 = 8028$ stones would be indeed enough.

Is it now possible to quit the things having less stones?

Speaking in same art and manner as earlier, we will try now to justify the following statement:

Sisyphus must take and have 4 stones of easiest and heaviest weight in any of his successful collections.

But there are many cases when he can reduce the number of taken stones in the intermediate weight categories. Namely, imagine for the sake of simplicity and exactness ***that the weights of all stones at the bottom of Acropolis are integers or multiples of the weight of the easiest stone*. All other stones have vanished away.**

Then already in the case of 3 different weights we might be able to spare some stones of intermediate weight. Namely we see that in the case of 3 different weights the collection of stones with weights

1, **1**, **1, 1, 2, 2, 3, 3, 3, 3**

is successful for Sisyphus.

Indeed if Hephaistos touches a pair (**1, 1**) of easiest stones then the only answer of Sisyphus would be another pair (**1, 1**) of other easiest stones. Similarly it will be with the pair (**3, 3**) of heaviest stones.

And now if Hephaistos touches the pair (**2, 2**) of the stones of intermediate weight then Sisyphus can answer with pair (**3, 1**) so sparing 2 stones of that intermediate weight **2 and vice versa** (**3, 1)** would be answered by **(2, 2**).

It seems further that there are no more essential cases to be considered.

So it seems that we are convinced that the collection

1, **1**, **1, 1, 2, 2, 3, 3, 3, 3**

is indeed the minimal collection for the case of 3 different successive weights 1, 2 and 3 and, what is slightly more general, that any minimal collection for 3 different weights must contain at least 10 stones.

Exactly in a similar way it can be proved that

In the case of 5 different weights the collection

1, **1**, **1, 1, 2, 2, 3, 4, 4, 5, 5, 5, 5**

is sufficient and that in the case of 5 different weights 13 stones is the minimal number to possess.

We kindly invite the reader to verify it. We guess that for this verification the equality

$$2 + 3 = 4 + 1$$

together with

$$3 + 4 = 5 + 2$$

would be employed as well.

BREAKTHROUGH XIX (XLIII). CONCERNING HIGH VALUE OF SIMPLE PROBLEMS OR A BRAVE BOY FROM GRADE 5 WILL DO EVERYTHING

In the Czechoslovakian Olympiad A.D. 1960/61 the following problem was proposed.

The sequence of numbers

1,2,2,3,3,3, 4,4,4,4,5,5,5,5,5…

is given.

What is the 1000th number in that sequence?

In order to set up access to all adventure or for the better involving to the process of solution we'll at first ask something what lays on hand:

What will be the 100th number in that sequence?

And only then something what is slightly more difficult, or:

What will be the 2007th number (of course!) in that sequence?

Looking at the given sequence we see that "numbers of that sequence are increasing and repeating itself more and more times".

So there was only one 1, but already two 2’s, further three 3’s, four 4’s, five 5’s and so on. In that place where 2007 will appear at first, afterwards it will be followed by other two thousand and six 2007’s.

So looking for the 100th number of that sequence

1,2,2,3,3,3, 4,4,4,4,5,5,5,5,5…

it is enough to take a sum

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + ... + n +\dots$$

and to “establish” with which n this sum firstly exceeds 100. This n with which this sum firstly exceeds 100 will be also exactly the 100th number of the initial Czechoslovakian sequence.

Taking the partial sums of the sequence

$$1 + 2 + 3 + 4 + 5 + 6 + 7 +\dots$$

we’ll get the well known “triangle” numbers

1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105,…

Because the first summand that brings that sequence beyond 100 is the 14th summand, it follows that the 100th summand of the initial sequence is 14. It might be also noticed that the 92nd , 93rd, …, 105th summands of the initial sequence are also 14.

And what are we going to undertake concerning the 1000th number of that sequence? Are we again going to count everything by hand?

No, we are not.

We gained some experience and we see that everything we need is to find such an n, that the sum of all integers from 1 till $n - 1$ is the “last” sum of consecutive integers which is still less than 1000, or

$$1 + 2 + 3 + 4 + 5 + 6 +\dots+ (n - 1) < 1000.$$

That means that for the sum of all consecutive integers from 1 till n

$$1 + 2 + 3 + 4 + 5 + 6 + \ldots + n > 1000.$$

In other words the latter sum is the “first” sum of consecutive integers starting from 1 up to n which exceeds 1000.

Doubling these sums we could write down both these conditions in the somewhat more convenient compact form as

$$2(1 + 2 + 3 + 4 + 5 + 6 + \ldots + (n - 1)) =$$
$$=(1+(n-1)) + (2+(n-2)) + \ldots + ((n-1)+1)=n(n-1) < 2000$$

together with

$$2(1 + 2 + 3 + 4 + 5 + 6 + \ldots + (n - 1) + n) =$$
$$(1 + n) + (2 + (n - 1)) + \ldots + (n + 1) = n(n + 1) > 2000$$

Concerning the order of magnitude of these two expressions or $n(n - 1) = n^2 - n$ and $n(n +1) = n^2 + n$ it could be told that they both are “not very far from n^2” (when being compared with n^2, of course) and lying on different sides of n^2.

We are looking then for such an n that n^2 be possibly near to 2000.

In our case the nearest square to 2000 is $2025 = 45^2$, because we could easily check that

$$44 \cdot 45 = 45 (45 - 1) = 45^2 - 45 = 2025 - 45 = 1980$$

and

$$45 \cdot 46 = 45 (45 + 1) = 45^2 + 45 = 2025 + 45 = 2070$$

so the wanted n appears to be 45.

Repeating almost word by word these considerations and changing only the numbers (preferably with a calculator in hand), we could also easily find what a number of initial sequence

1,2,2,3,3,3, 4,4,4,4,5,5,5,5,5…

resides in the 2007th place and which one in the 2008th or even in the 2009th place. No essential difficulties are to be seen also for determining the 1 000 000th or even 1 000 000 000th or any other member of that sequence.

BREAKTHROUGH XX (XLIV). FUNNY MATHEMATICS

All of us at least 10 times in the life have heard the aria from the opera of Giaccomo Puccini "Figaro there, Figaro here". It's about the barber who was able to arrange many matters and in the same time was eager to undertake practically everything you could ever imagine. Using modern terminology we would say that he was remarkably flexible and was eager to answer every challenge. Great was his motivation for any action.

There is also a lot of hidden flexibility everywhere in mathemtics or even in the simplest arithmetics. Every variable, independently how is it denoted, is the first good example and source for this. Everything is in move, is changing and looking for optimal place or greatest possible value. Similarly it is also in our everyday life and everywhere under the sun.

First atomar example of flexibility is probably got by taking any two subjects A and B and afterwards changing their places so getting (B, A) from (A, B). Changing their places we'll get rather similar but not the same reality (B, A).

Needless to explain that Mr President and Mr Prime Minister is not quite the same as Mr Prime Minister and Mr President.

These who could have some doubts may compare U.S.A. and United Kingdom together with Germany.

No doubt that these pairs are build from the same elements A and B, but their places in these pairs are different. For example it might happen that writing (A, B) we might mean that A as written first is almost more important than B written in second place, or vice versa.

It might also happen that A and B are some statements and we might assume and define that writing (A, B) we have in mind that A implies B or "from A follows B". Then of course (A, B) might be different from (B, A) because if A implies B then not necessarily B implies A. If it were so then we would have much more equivalent conditions as we actually have.

Let us regard some statement A of poetical nature "The sun appeared in the sky" and another statement B also from an everyday life context "It isn't dark in the kitchen". Then it is clear that A doesn't imply B. That has a simple human sense. A implies B would mean that if the sun appeared in the sky then necessarily it is not dark in the kitchen.

But that's not necessarily so. The kitchen can be even without windows or it might also be that there are some but they remain either always or sometimes closed. And we could have a situation with the sun high in the sky and with the darkness in the room with closely shutted windows.

This ought to be rather natural example that A implies B isn't right in all cases.

If we would take another sometimes more realistic statements with more usual circumstances then it might be that the statement A implies the statement B.

For example, if the statement A is:

"The sun appeared in the cloudless sky"

and the statement B sounds

"It isn't dark in the room with opened windows"

then the fulfillment of A implies also the fulfillment of B but not vice versa because it might be that it is a midnight – so that the condition for A is not fulfilled – but in that room with still opened windows we switched on the electricity – so that B is fulfilled.

But B being fulfilled and A being not demonstrates that B does not imply A.

At the end we would cite an old joke about the Sun and the Moon where the question is: which of them is more important?

The answer is: the Moon is more important because the Moon is shining during the night when there is no Sun in the sky, while during the daytime it's allways bright simply so.

BREAKTHROUGH XXI (XLV). MACAVITY THE MYSTERY CAT OR CONTINUING FUNNY MATHEMATICS

Macavity's a ginger cat, he's very tall and thin:
You would know him if you saw him…

(T. S. Eliot)

There is a very famous Latin expression which states that we all are changing with the time which in Latin sounds so impressive that for any person skilled in English practically no translation is needed –

Tempora mutandur et nos mutamur in illis.

The following funny history will try to demonstrate that not only human beings are changing with the time but the famous Macavity also.

As a proof of these changes we will tell how did he propose a contract to the suitable department of Scotland Yard. This was intended to be a contract where the dignity of both sides was carefully discussed and fully guaranteed. The sum of contract and other interesting circumstances were being held in great secret – nevertheless some information as usual became known to the public opinion and some comments were distributed. Still it wasn't a scandal but rather some kind of disappointing news.

The famous journalist Mr. Stanley cleared that the main sum of contract was due to and based on Macavity's guarantee to inform about the rain one day before it would start.

Speaking more exactly Macavity took up the following obligation: he promised to communicate about the rain falling in the yard of Scotland Yard by sneezing a day before.

Many people claimed that even Macavity cannot give the necessary guarantees by sneezing a day before it will be raining in the yard of that famous Scotland Yard.

Mr. Stanley himself initiated that discussion by his famous statement saying: imagine Macavity has just sneezed. Does it follow that tomorrow it will be the rain in Scotland Yard? His answer was:

No, it doesn't. Macavity could have sneezed because he catched cold and not because of rain.

Another circumstance demonstrating that the wicked cat could simply fulfill the contract by sneezing every day – and the famous condition of contract concerning the guaranteed information about the coming rain in such a case would be never violated.

We would like to finish this chapter not with an instruction for the possible reader of the kind: please take some care when examining and verifying what implies what, but rather with the following optimistical lines:

"If all the world were apple pie,
And all the sea were ink,
And all the trees were bread and cheese
What should we have to drink?"

At that place we would like to say that all the time long in these two chapters we were speaking about the matters similar to that one. Imagine that the following holds: if I'm able to translate that sentence correctly, I would be happy. That's nice. Imagine and believe that I'm happy now. Unfortunately for me as a translator it doesn't mean that this is because I have translated correctly this sentence. It is possible that I'm happy exceptionally from the reason that I have an everlasting hope to translate some sentence correctly in the future.

We promised to the reader to conclude that Chapter with citing of some poetical lines with a clear mathematical or human (we know you know that in successful case this is the same) idea. We've found the lines that express the reality that *B* can be contained in *A* but the converse is not right - that is, *A might not be contained in B* - even showing who is responsible for that.

With these lines or the following epigram we are indeed completing the given breakthrough:

Sir, I admit your general rule,
That every poet is a fool:
But you yourself may serve it
That every fool is not a poet.

It’s a pity of course for many of us that we aren’t the authors of these lines. These classical lines belong to *Matthew Prior* (1664-1721), an English diplomat and poet. He was a master of what Addison called “the easy way of writing”, that is, the light epigrammatic verse. Reading these lines we understand better what these lines could mean.

Not in vain the ancient people repeated

Qui bene cantat, bis orat

what corresponds to

To sing nice is to pray twice.

Double done is best done.

BREAKTHROUGH XXII (XLVI). IS THERE INDEED A LIFE ON MARS OR ABOUT SURPRISES IN STATISTICS AND BOOKKEEPING INDUSTRY

If anybody were going to say to you that there is a life on the planet of Mars you would indeed be at least a bit astonished. This theme possesses a long history. That history in a lucky way is a history without the history.

In these long lasting days in the former Soviet Union all shifts of population knew the nice song with the words “as it is stated by space shipmen and day-dreamers that on Mars the apple-trees will blossom”.

So it is quite clear what was just stated. Hearing that there is a life on Mars what could you say **pro and**

contra? Of course you could say many many things. For example you could ask, why especially Mars?

Why not Mercury?

Let us give the full freedom to our fantasy.

It occurred that there is a life on Mars and on Mercury. Moreover it became known that all these inhabitants on both planets – ladies and gentlemen in equal way – have a honourable duty to learn all their life along. Their knowledge is measured by numbers from 1 till 10 similar as in so many countries on our Earth.

Still studying their Science Report A.D. 2007 some strange things occurred. Because fax reports from these planets are still very expensive they send only an average estimation of each of their inhabitants.

So investigating these materials we came across to some facts that by opinion of experts are highly contradictory.

Below we cite them all.

1. The average estimation, *which since then will be called **average mark**, of all Mercury's ladies is higher than the average mark of all Martian ladies.*

2. *The average mark of all Mercury's gentlemen is higher than the average mark of all Martian gentlemen.*

3. *The total **average mark** of all Mercury's inhabitants **is not higher** than the average mark of all Martians – but it **is less**. It is so paradoxical that we will repeat this once again.*

*The total **average mark** of all Mercury's inhabitants **is less** than the average mark of all Mars's inhabitants.*

Honourable reader! Are you able to believe in it?

1. Are you able to believe that it might happen that Mercury's ladies are learning ***better*** than the ladies on Mars?

Very probably you are. Once we are better (usual state of wellness), once our neighbours are.

2. Are you able to believe that it might happen that the Mercury's gentlemen have also ***better*** science achievements then their colleagues on Mars?

Very probably you still are.

3. Can you imagine and are you able to believe that it might happen that under conditions 1 and 2 the whole Mercury's human population has ***worse*** average science mark than the whole human population on Mars?

No, hopefully you are not able to believe in it. You would say that this is incredible; you would say that this is unbelievable. You would say that this is unprobably and that this can never take place.

You are essentially right. In most cases if our ladies are better and if our gentlemen are better then we all together are also better.

Yes, in most cases it is so.

But "in most cases" doesn't mean "in every case".

Here you'll see an example.

If I crossed the sea successfully 1000 times it's so nice. If I crossed successfully the sea 1000 times then highly probably I will do it with the same success also in the 1001st crossing. Highly probably doesn't mean surely. Highly probably doesn't give me full guarantee. The 1001st crossing might be my last crossing which ever takes place.

Our example as you will see is not at all complicated and involves only six persons from both planets. Three of them are representatives of Mercury and three of them are from Mars. For the sake of complicity we list them all: the only Mercurian Lady and Gentlemen 1 and 2 (shortly MeL, MeG1 and MeG2) and Martian Ladies 1 and 2 and the only Gentleman (or shortly MaL1, MaL2 and MaG).

It happened that their average marks are the following: Mercurian Lady – 10, Mercurian Gentlemen 1 and 2 – mark 6 for both, Martian Ladies 1 and 2 – mark 9 for both and Martian Gentleman – mark 5.

So now we all, the inhabitants of the Earth, are eagerly studying the table of their achievements:

Table for MERCURY

CODE	*Mark*	*Ladies' average*	*Gents' average*	*Total average*
MeL	***10***	***10***		
MeG1	***6***		***(6+6)/2 =6***	***(10+6+6)/3=7.33***
MeG2	***6***			

together with the corresponding

Table for MARS

Code	*Mark*	*Ladies' average*	*Gents' average*	*Total average*
MaL1	***9***	***(9+9)/2=9***		
MaL2	***9***			***(9+9+5)/3=7.67***
MaG	***5***		***5***	

This is of course paradoxical from the one side, but on the other hand regarding all these data again we understand that although the average mark of both Martian ladies is 9 and so is less than the average mark of the only Mercurian lady which is 10, still both Mercurian ladies together gathered $9 \cdot 2 = 18$ points and that is

$$18 - 10 = 8$$

points more than the only Martian lady did. These plus 8 points will soon play the main role in the drama that average mark of all Marsians is higher that the average mark of all Mercurians. Let's continue our observations.

Both Mercurian gentlemen with average mark 6, which is greater than the average of the only Martian gentleman which is 5, gathered only $6 \cdot 2 = 12$ average mark points, and that is only

$$12 - 5 = 7$$

points more than the only Martian gentleman did. Difference

$$8 - 7 = 1$$

which now is in favour of the Martians demonstrates and proves also that the Martians as a whole population proved to be more successful than the whole Mercurian population.

BREAKTHROUGH XXIII (XLVII). LOVES, DOESN'T LOVE OR PREFERENCES AND THEIR MODELS

We are hundred percent sure that every notable life phenomenon possesses its mathematical analogue.

The first and very deep arithmetical model for the situation "loves, doesn't love" is due to the correspondence when "loves" corresponds to 1, "doesn't love" corresponds to 0 and so we have a human link to the whole even-odd models. This is so when we are dividing the whole set, which we are to consider, into two parts afterwards selecting one of them. Doing this we are very often able to achieve a remarkable progress or at least get some impulse for our deeds. The fruits of it we enjoy practically every day.

Consider some adopted problem when three brothers Andrew, John and Justus with a clear social(ist) orientation were being involved in some unusual privatization process.

We will now report all this with necessary details.

A 7×7 square table with 49 entries is given. In each entry we have a different integer from 1 till 49. Now we see the oldest brother John sitting in one room and carefully examining all the numbers in that 7×7 table. His brothers Andrew and Justus are in the other room so that they can hear perfectly what John is saying to them but they are not able to see the numbers.

Now John is providing the idea of privatization of the sums of rows and columns of the given table. We understand that a square table 7×7 has 7 rows and 7 columns with 7 integers in each row and in each column.

So the following process runs. John is taking each row one by one and every column also one by one carefully adding up all its numbers in each row and column, distributing these sums as follows: even sums are given to Andrew and odd ones to Justus. They are writing down each sum they get. After all these 14 sums were distributed each of them calculates the sum of numbers he'd got. Then they are going to compare their sums (in fact they are comparing the sum of sums). The brother with the greater sum will get a 1000 Euro money price.

And actually they completed their summings and calculations, and in an astonishing way it turned out that these both super sums of Andrew and John appeared to *be the same. Only now they noticed that such a case is not mentioned in the awarding instruction.*

What to do? Why that case of equal sums wasn't regarded in the awarding instruction? Is it impossible? Did any of them do something wrong? It is known that John is a master in taking sums. He never fails.

They were regarding the situation and waiting for their father as the biggest authority of taking the sum in that family.

Now the father just arrived. They reported him the situation. They repeated to their father: we have taken the 7 x 7 table with 49 entries with all different integers from 1 till 49 in each entry. Then John calculated all sums in each of 7 rows and in each of 7columns. Then the even sums went to Andrew and the odd ones went to Justus. Finally Andrew calculated his super sum E and Justus – his super sum O. It appeared that

$$E = O$$

Father, asked they, we do not know what to do? We are not instructed for such a case.

Father thought for a while and the flame of sadness appeared in his eyes. He didn't say any word but the brothers understood that something had happened.

What's happened?

Father who was skilled in pedagogic ordered them to find out what's wrong. We remind that John makes no mistakes in taking sums, he never fails doing that.

After some considerations brothers phoned to their Granddad who was the teacher for Mathematics and some other even more serious things.

The Granddad almost immediately made the following observations:

1. Even sums landed by Andrew mean that also his super sum E is even as well (adding even numbers to even ones it is impossible to get an odd number).

2. In that case ***E = O*** indicates that also the super sum of Justus is even.

3. The sum of these two equal even super sums as a sum of any two even numbers

$$\mathbf{E} + \mathbf{O} = 2\mathbf{O} = 2\mathbf{E}$$

is clearly divisible by 4.

4. In that case the sum of two super sums **E + O** is such a sum where each number of the initial 7 x 7 table (containing all the integers from 1 till 49) is counted exactly twice – once in the row and once in the column. In other words, this sum of super sums is the double sum of all integers from 1 till 49:

$$2\,(1 + 2 + 3 + \ldots + 47 + 48 + 49).$$

So if all brothers did everything right then this double sum must be divisible by 4.

5. Then the “single” sum

$$1 + 2 + 3 + \ldots + 47 + 48 + 49$$

is divisible by 2 or is even.

6. The sum

$$1 + 2 + 3 + \ldots + 47 + 48 + 49 =$$
$$= (1 + 49) + (2 + 48) + \ldots + (24 + 26) + 25 =$$
$$= 50 + 50 + 50 + \ldots + 25 = 50 \cdot 24 + 25 = 625$$

is not even, giving the contradiction to what was being stated in “5”.

It indicates that the situation you’ve got – so he spoke to his grandchildren – is completely impossible. So at least one of you must be wrong.

And now the Grandma silently entered the room and was carefully listening to what was being said. She added observations 8, 9, 10 which were formulated by her own and sounded as follows:

8. My dear Grandson John doesn't ever make mistakes by taking sums. So John can't be wrong also in that process.

9. Because John is right but there is a mistake which was made then it follows that either Andrew or Justus in an astonishing way made an arithmetical mistake. Nevertheless they both are great because they understood it.

10. They are also noble because they are not going into details which of them made a mistake. Alone from that reason they both are worth at least the tablet of chocolet.

After such objective but also idyllic and sentimental end of these reflections it should be added that we have discussed and solved the Problem 3 [vide 3] from the International Team Contest „Baltic Way“ A. D. 2001, which took place in Hamburg (Germany).

BREAKTHROUGH XXIV (XLVIII).
AFTER YOU'VE STARTED COMPLETE IT ANYWAY

Clearly such an instruction is a proverb in any language. This is perfectly true and nice but at the same time it is easier said than done. For example, you can't find a person, which didn't know that to be eager to work or to be industrious or diligent is very useful and honorable. But being at least quite a bit objective we must confess that it happens even with us: we loose our will and some things which we'd undertaken remain, mildly speaking, uncompleted.

It might always be otherwise with the reader, our congratulations for you in any case. But it happens sometimes, although not to often, with the author of this manuscript.

In order to demonstrate that to complete the things is the aristocracy in action, we'll tell you about problem which was proposed in some competition for bright minds in Lithuania A.D. 2005.

Here is its laconical formulation - one sentence only:

For what N is it possible to divide the set of all integers from 1 till N into 3 disjoint subsets with the same sum of numbers in each subset?

It is clear that having such a wish we must have at least 3 elements in our initial set so that $N \geq 3$.

Further on we repeat the question for the set $\{1, 2, 3, 4\}$ and ask us whether it is possible to divide it into three disjoint subsets with the same sum of elements in each subset. The answer is resolutely **no** because if it was possible then the sum of all 4 numbers

$$1 + 2 + 3 + 4$$

which is 10 must be divisible by 3, but clearly it isn't.

Of course that consideration employing the impossibility of expressing the sum $1 + 2 + 3 + 4$ in the form of three equal integer summands corresponds to the level of grade 8 or 9 or the high-school level.

But that problem can be given to constructive girls and boys already in primary school where the notion of divisibility isn't repeated every day.

In such a primary school at any constructive level it is possible to do everything simply regarding all possible cases of splitting the set

$$\{1, 2, 3, 4\}$$

into three disjoint subsets with at least one element, because otherwise their sums of elements couldn't be equal.

So we can simply list all possible partititons of the set {1, 2, 3, 4} into 3 subsets each of which contains „something“. Because, as stated, each of these 3 subsets contains „something“ from the initial subset of 4 elements it means that one of these 3 subsets contains exactly 2 elements and remaining 2 – exactly one element each. But one subset containing 2 elements from the 4-element set may be taken in 6 different ways fully determining also the partition in question. The list of the possible partitions will be as follows (the set containing 2 elements is written as the first in each figure brackets):

{(1, 2), (3), (4)}, {(1, 3), (2), (4)}, {(1, 4), (2), (3)},
{(2, 3), (1), (4)}, {(2, 4), (1), (3)}, {(3, 4), (1), (2)}.

We state that in no case the sums of all three disjoint subsets of the given partition of the set {1 ,2, 3, 4} are equal.

Such practise shows and demonstrates us not only that we are right in what we've stated, but even slightly more, e.g. looking at the list of partitions we would also answer the following possible question:

In how many ways is it possible to split the 4-element set into 3 non-empty subsets so that the sum of elements of some two of these 3 subsets would be equal?

The answer is 2.

From that list of all possible splittings some other related question may be exposed and the answers promptly presented, e.g.:

What is the probability that splitting the 4-element set into 3 non-empty subsets the sum of elements of some two of these 3 subsets will be equal?

The answer is 1/3, namely, 2 of 6 possible splittings.

Remark. The partition of the set into three non-empty subsets could be seeked for in a mechanical way with no tension and emotions.

It could also be seeked for in some psychologically probably more convincing way, namely, when splitting the initial set into 3 disjoint subsets the smallest number 1 couldn't be left alone because of equality of sums.

So 1 must be accompanied by some other element.

If 1 is accompanied by 2 and other subsets are non-empty then we would get the partition

$$\{1, 2, 3, 4\} = \{1, 2\} \cup \{3\} \cup \{4\},$$

which gives us nothing because the sums in subsets are not all equal.

If 1 is accompanied by 3 and other subsets are non-empty then we would get another partition

$$(1, 2, 3, 4) = \{1, 3\} \cup \{3\} \cup \{4\},$$

which is again not suitable because the sums are again not equal.

Finally, if 1 is accompanied by 4 and other subsets are non-empty then we would get another partition

$$\{1, 2, 3, 4\} = \{1. 4\} \cup \{2\} \cup \{3\},$$

which is also not suitable because in no two of these 3 subsets the sums of elements are equal.

After that experience it might also be stated that no set containing 4 different numbers couldn't be splitted into 3 disjoint subsets with equal sums of elements because then some 2 subsets of that splitting must be *one–element subsets so their sums are not equal – the initial integers were all different!*

Further simple related questions could be:

1. *Is it so with the set containing any 5 different integers?*

2. *Find a set containing 5 integers with the smallest possible sum of integers admitting the desired partition.*

BREAKTHROUGH XXV (XLIX). CONTINUING SPLITTING THE SETS OF FIRST INITIAL INTEGERS

We remind to the reader that we failed in our attempts to split any subset containing 4 integers into 3 disjoint subsets with the same sum of elements in each subset.

Now we are going over to the sets with more elements. The next set which we intend to split into 3 parts with equal sums of elements is the 5-element set {1, 2, 3, 4, 5}.

In that case we are immediately successful because of the possible partition

$$(1, 2, 3, 4, 5) = (1, 4) \cup (2, 3) \cup (5)$$

with clearly equal sums of numbers in all subsets.

This also promptly answers both questions 1 and 2 from the previous chapter.

So our first success is noted in the case $N = 5$ when dealing with the set {1, 2, 3, 4, 5}.

The following step would be the case $N = 6$ or splitting the set {1, 2, 3, 4, 5, 6} into 3 disjoint subset with equal sums. This attempt is successful again because of

$$\{1, 2, 3, 4, 5, 6\} = \{1, 6\} \cup \{2, 5\} \cup \{3, 4\}.$$

The next case with the set {1, 2, 3, 4, 5, 6, 7} seems to be similar to the case with the set {1, 2, 3, 4} or seems to be unsuccessful. The remarkable difference now is that if we will start again considering all possible splittings „by hand“ then we would face the prosaic possibility to omit exactly that case which happens to be exactly the case for which we're looking for.

We can spare that troubles and considerable amount of effort and energy again by noticing that the sum of numbers in the set

$$\{1, 2, 3, 4, 5, 6, 7\}$$

or the sum

$$1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$$

is not divisible by 3 so that **every attempt to split the set {1, 2, 3, 4, 5, 6, 7} into 3 disjoint subsets with equal sums of elements will always fail.**

Comparing the next set {1, 2, 3, 4, 5, 6, 7, 8} with the first successful case {1, 2, 3, 4, 5} we see that these sets differ by 3 numbers 6, 7 and 8 with the sum of „new“ elements $6 + 7 + 8 = 24$ clearly divisible by 3. The average of these 3 newcomers is

$$(6 + 7 + 8) : 3 = 7.$$

So we begin to feel that it would be possible to modify for it the „old" splitting

$$\{1, 2, 3, 4, 5\} = \{1, 4\} \cup \{2, 3\} \cup \{5\}.$$

Everything we need for success in that modifying of the „old“ splitting into a „new“ one is to achieve an increase of 7 in each subset of splitting.

That is possible to achieve acting as follows:

1. Extract 1 from the subset where it was contained and replace it by 8 (total increase of sum in that subset will be exactly 8 – 1 = 7 as planned).

2. Include 7 into the next subset of that splitting (total increase is also 7 as required).

3. Add 6 and the „free agent“ 1 to the last set of splitting (total increase again would be 6 + 1 = 7 as planned and required).

So from the „old“ splitting

$$\{1, 2, 3, 4, 5\} = \{1, 4\} \cup \{2, 3\} \cup \{5\}$$

we've got a „new“ splitting

$$\{1, 2, 3, 4, 5, 6, 7, 8\} = \{4, 8\} \cup \{2, 3, 7\} \cup \{1, 5, 6\}.$$

But exactly in the same way we could also proceed having initially the splitting of the set $\{1, 2, 3, 4, 5, 6\}$

$$\{1, 2, 3, 4, 5, 6\} = \{1, 6\} \cup \{2, 5\} \cup \{3, 4\}$$

and wishing to enlarge it by next three elements 7, 8, 9 extending that „old“ partition into the „new“ one and, of course, not losing the equality of sums in all subsets. Repeating everything word by word or again taking 1 out of first set of partition and replacing it by 9, then adding 8 to the second subset and finally adding 7 and the actually „free agent“ 1 to the third set of our splitting we would get the partition of the extended set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ in the form

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9\} = \{6, 9\} \cup \{2, 5, 8\} \cup \{1, 3, 4, 7\}.$$

But the most impressive thing what was just achieved is the following one: having any partition or splitting of the set $\{1, 2, \ldots, N\}$ into three disjoint subsets P1, P2 and P3 (P1 contains 1) with equal sums of elements and using the same extension procedure which was applied already twice we *can easily extend any splitting of the given set to the set which is enlarged by* *three successive elements* $N + 1$, $N + 2$, $N + 3$, describing explicitely the partition of extended set $\{1, 2, \ldots, N, N + 1, N + 2, N + 3\}$ into three subsets with the same sum of elements in each subset again and getting

$$\{1, 2, \ldots, N, N + 1, N + 2, N + 3\} = $$
$$= ((P1/ \{1\}) \cup \{N + 3\}) \cup ((P2) \cup \{N + 2\}) \cup$$
$$\cup ((P3) \cup \{1, N + 1\}).$$

Repeating this we have the possibility to get infinitely many other extended partitions from the given partition getting something what might be called “an approach to infinity” or “unbounded growth” what is not always easy and not always possible to realize.

Summarizing we state:

1. If N is 1, 2 ,3, 4 then the splitting of the set {1, 2, …, N} into three disjoint subsets is obviously impossible.

2. If N = 5, then the splitting into three sets with equal sums is possible – as it was announced but still not written – namely

$$\{1, 2, 3, 4, 5\} = \{1, 4\} \cup \{2, 3\} \cup \{5\}.$$

The procedure of extension adjoining three succesive „next“ integers which we’d described and applied several times demonstrates the possibility to do it also in the cases N = 8, 11, 14, …, 3M + 2… .

3. If N = 6, then the splitting is again possible:

$$\{1, 2, 3, 4, 5, 6\} = \{1, 6\} \cup \{2, 5\} \cup \{3, 4\}.$$

The same procedure of extension provides the possibility to do it for N = 9, 12, 15, …, 3M,….

4. If N = 7, then this is as stated impossible because the sum of all elements or

$$1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$$

isn’t divisible by 3. But the sum of elements remains indivisible by 3 after any extension of it by adding any 3 *successive integers because, extending any set by 3 successive integers, the sum of these „newcomers“*

$$N + 1, N + 2, N + 3$$

is 3N + 6 = 3 (N + 2) and is divisible by 3.

But then the sum of all elements of the extended set isn’t divisible by 3 as well. So we repeat again that the partition of any set with the sum of elements not divisible by 3 into 3 disjoint subsets with equal sums of elements is impossible.

By the way the splitting would be also impossible if we wished to adjoin any number of elements with the sum of elements divisible by 3.

So in the case N = 7 and consequently also in all the cases 10, 13, 16, ... , 3M + 1,… such splitting isn’t possible.

We conclude the Chapter with the final remark consisting of two parts (A) and (B) the importance of which for the logical life seems to be immense:

(A) If the sum of elements of any set S of integers isn’t divisible by 3 (or by any other number K as well) then it’s not possible to splitt the set S into 3 disjoint subsets (into K disjoint subsets) with the same sum of elements in each subset.

(B) If the sum of elements of any set S of integers is divisible by 3 (or by any other number K as well) then this in no way implies or guarantees the possibility to split the given set S into 3 disjoint subsets (into K disjoint subsets) with the same sum of elements in each subset.

An example for such impossibility could be, for instance, the set S consisting of 6 elements

$$S = \{1, 11, 111, 1\,111, 11\,111, 111\,111\}$$

with the sum of elements

1+ 11 + 111 + 1 111 + 11 111 +111 111 = 123 456

being divisible by 3.

If you are brave enough not to give up the hope that it is possible to divide that set with only 6 elements into three subsets with the same sum of elements in each then think where the greatest number

111 111

would be located.

BREAKTHROUGH XXVI (L). ANOTHER NICE PROBLEM WITH EQUAL SUMS

"You boil it in sawdust: you salt it in glue:
You condense it with locust and tape:
Still keeping one principal object in view –
To preserve its symmetrical shape."

To divide something into parts preserving some sort of equal rights of these newly divided parts (we'd just got) might be neither easy nor simple. Even in these somewhat simplest cases when we are arranging "2 from 1" some tensions and complications might be met at each step.

Still in most cases alone from the psychological (human) point of view the ability to split or peacefully divide something into parts preserving equality of any kind and nature brings some satisfaction.

The following example should illustrate similar wishes and hopes. We will formulate it as a question in order that we all would be able to look for the possible right solution as an answer to that exposed question. So we are raising bravely the following

Question. Is it indeed possible to divide the set of all integers from 1 till 105 into 15 subsets with 7 numbers in each subset in the way that the sum of all numbers in every subset is the same for each subset?

We will start again arranging some small things or making almost unnoticeable steps. Doing something of the kind it is of great importance to believe that it is possible to achieve what you are going to. Because doing something what is not completely trivial you will face some difficulties, and difficulties and doubts usually run in pairs.

These difficulties are connected mainly with the essence of the given problem – one could say that

Difficulties are living mainly in the problem itself, while doubts – mainly in our heads.

So really believing that such a partition of the set of first successive 105 integers

$$\{1, 2, 3,\ldots,103, 104, 105\}$$

into 15 parts containing 7 elements with the equal sum of numbers in each is possible we'll proceed computing the sum of elements in each subset "splitting 1 into 15" .

The sum of all first 105 successive integers possesses 105 summands, the mid summand being 53. So

$$1 + 2 + 3 + \ldots + 103 + 104 + 105 =$$
$$((1 + 105) + (2 + 104) + (3 + 103) +\ldots+ (52 + 54) + 53)=$$
$$= (106 + 106 + 106 +\ldots+ 106 + 53) = 106 \cdot 52 + 53 =$$
$$= 53 \cdot 2 \cdot 52 + 53 = 53(52\cdot 2 + 1) = 53 \cdot 105 = 5565.$$

Then the sum of numbers of these subsets with 7 numbers in each must be

$$5565 : 15 = 371.$$

Now when we are more conscious what we intend to achieve **we are doing our first mild shift:** we move our whole set of 105 first successive integers (with the number 53 as the mid number) to the left by 53 units so that the mid number 53 after shifting becomes 0 and the shifted set itself is now

$$\{-52, -51, -50, \dots, -1, 0, 1, \dots, 50, 51, 52\}.$$

What is better now? What do we achieve this way?

We achieve that our set is symmetrical with respect to zero now. That is, our set now possesses the precious symmetry meaning that if any number x is in our set then the opposite number (-x) is also an element of our set.

Each of us has rather strong feeling of symmetry and remarkable experience of dealing with it under various circumstances.

It could be noticed that the initial set S was also symmetrical with respect to 53, that is, if any element x of it, say 77, is contained in our set and lying (as in our case) on the right side from the mid number 53, then the number lying on the other side of 53 within the same distance from the number 53 as x must be also contained in S.

In our example the distance of 77 from 53 is $77 - 53 = 24$ and the number on the other side with distance 24 from the mid number 53 is, of course, $53 - 24 = 29$.

(The technical guarantee that we performed well and our calculations are correct is the equality $(77 + 29)/2 = 53$.)

So now we’ve reduced our problem to the following one:

Is it possible to split the set of 105 succesive integers

{-52, -51, -50, ... , -1, 0, 1, ... , 50, 51, 52}

into 15 subsets with 7 elements in each subset and with the sum of integers 0 in each of them?

The following also very useful move might be the following one: we are eager to find 15 subsets with only 3 integers in each subset and with a sum of all 3 integers in each being also 0.

In our epigraph it stood: *Still keeping one principal object in view/to preserve its symmetrical shape.*

In our case it means that the set of all 15 · 3 = 45 involved numbers is supposed to be symmetrical with respect to 0.

(That move will be last but one as the reader will just see.)

In that intermediate construction 15 · 3 = 45 integers from these 105 new integers we've got will be employed and the set of these newly taken 45 numbers will be also expected to be symmetrical.

The construction we'll provide now will be very prosaic: we'll simply list all these 15 subsets each subset containing 3 integers as promised:

{-1, 0, 1}, {4, 5, -9}, {- 4, -5, 9}, {7, 8, -15}, {-7, -8, 15},
{10, 11, -21}, {-10, -11, 21}, {13, 14, -27},
{-13, -14, 27}, {16, 17, -33}, {-16, -17, 33},
{19, 20, -39}, {-19, -20, 39}, {22, 23, -45},
{-22, -23, 45}.

It might be seen directly that the listed set contains 45 integers and is indeed symmetrical with respect to 0. Consequently, the remaining set of not yet taken integers containing remaining 105 – 45 = 60 of them is also symmetrical with respect to 0 as any difference of two symmetrical sets.

Moreover, any set which is symmetrical with respect to 0 and doesn't contain 0 can be splitted into disjoint subsets each subset being of the form $\{-x, x\}$.

And now the final step of the construction of 15 sets with 7 elements in each and with sum 0 in each. We simply take any two sets of the form $\{-x, x\}$ and include them into each of these 15 subsets already listed above.

Now shifting the set back to the right by 53 units we get the partition of the initial set of first 105 integers with 7 elements in each subset and with the same sum of numbers in each subset (being $371 = 53 \cdot 7$ as it was already indicated).

We ask the reader now to compare it with the problem from International Mathematical Olympiad in Braunschweig, Germany, 1989, or confront what we've done with the problem from World Cup of High-school students in Mathematics.

Prove that the set {1, 2, ..., 1989} can be expressed as the disjoint union of subsets A_i *(i = 1, 2, ..., 117) such that*

(A) each A_i *contains 17 elements;*

(B) the sum of all the elements in each A_i *is the same.*

BREAKTHROUGH XXVII (LI). DOMINO STONES IS ALSO A NICE GAME TO PLAY

But the Barrister, weary of proving in vain
That the Beaver's lace making was wrong,
Fell asleep and in dreams saw the creature quite plain
That his fancy had dwelt on so long.

In the nineties in the USA there existed a challenging quarterly for mathematics “Consortium” whose name is clearly of Greek-Latin (probably more Greek than Latin) origin.

In “Consortium” we’ve found a problem which is equally simple, challenging and accessible. Its accessibility could be compared with that of Sudoku.

For the sake of completeness let’s remind the reader what a wonder Domino stones are. In the quasi-mathematical terms it could be told that the domino stone is a rectangular of size 1×2 “made” from 2 “unit” squares with some number between 0 and 6 inscripted in each of these unit squares. Otherwise or speaking more scolarly language the given domino stone or piece could be identified with some unordered pair of numbers (K, L), where K and L independently can take any of possible values 0, 1, 2, 3, 4, 5 or 6. Unordered pair, speaking understandably, means that we make no difference between pairs (K, L) and (L, K).

After that remark we might simply list all possible domino stones as such (unordered) pairs in the following form one by one preserving some understandable order:

(0; 0), (0; 1), (0; 2), (0; 3), (0; 4), (0; 5), (0; 6),
(1; 1), (1; 2), (1; 3), (1; 4), (1; 5), (1; 6),
(2; 2), (2; 3), (2; 4), (2; 5), (2; 6),
(3; 3), (3; 4), (3; 5), (3; 6),
(4; 4), (4; 5), (4; 6),
(5; 5), (5; 6),
(6; 6).

As we see there are exactly 28 domino stones.

It could and ought to be noticed that in all these stones each number from 0 till 6 appears 8 times in exactly 7 stones. It’s nothing astonishing in “8 times in 7 stones” because one stone is “double” with the same numbers on both sides of it, and on the remaining 6 stones each number from 0 till 6 appears exactly once together with every other of remaining numbers. It may be seen from our list.

Because, as it was counted and listed and noted, there are exactly 28 stones with every stone “made” from some 2 unit squares, for the complete domino set $28 \cdot 2 = 56$ unit squares are needed. By the way, 56 may be arranged in the form of rectangular 7×8.

For the emotional refreshment this problem, which is, as noted, taken from Consortium, will be presented as a part of scientific dream of some brave soldier who was once had been a bright boy and whose name is John Brown. Some thrilling terminology and some flavour of nightmares may also be met.

So once John Brown also “fell asleep and in dreams saw creature quite plain”, that creature being the desk of size 7×8 completely covered with the set of all domino stones lying on the table which was on the deck of some transatlantic ship. Afterwards John remembered only the notion of storm and the tremendous wave that came over the desk. Afterwards John stated with some surprise: after wave went by that desk remained just as it had been, the domino stones also. The only difference with what’d been before was that there were no boundaries to be seen between these domino stones. These numbers in domino stones formed now precisely the table of size 7×8 filled by the numbers 0, 1, 2, 3, 4, 5 and 6, each of them being seen in that table 8 times.

Then in the somewhat strange way the magician Dumbldor occurred and announced to the brave John Brown who listened with some astonishment and ever-growing surprise, that he will cause him no harm, and also added that he would be able to get rid of that mare if he will be able to recover the previous boundaries of all stones. After saying this Dumbldor disappeared and John waked up in such a state as if nothing had happened. The only thing reminding him that something had taken place was the clear image of the configuration of the stones in his head. He had written it down at once in the very moment.

It was the following table:

5	**5**	**5**	**2**	**1**	**3**	**3**	**4**
6	**4**	**4**	**2**	**1**	**1**	**5**	**2**
6	**3**	**3**	**2**	**1**	**6**	**0**	**3**
3	**0**	**5**	**5**	**0**	**0**	**0**	**6**
3	**2**	**1**	**6**	**0**	**0**	**4**	**2**
0	**3**	**6**	**4**	**6**	**2**	**6**	**5**
2	**1**	**1**	**4**	**4**	**4**	**1**	**5**

Is it indeed possible to recover the boundaries of all domino stones of the complete domino set?

BREAKTHROUGH XXVIII (LII). AGNIS ANDŽANS AND HIS CONSORTIUM PROBLEM

In this Chapter we will present and with the satisfaction propose for your attention and pleasure the problem due to well-known Latvian mathematician, problem composer and specialist for mathematical challenging and creativity and many other areas Agnis

Andžāns. Professor Andžāns had been training the Latvian mathematical team for many decenniums. He is also a teacher of many Lithuanian gifted students, e.g. by taking part on Lithuanian National High-School Assembly.

Hereby we present that problem once also published in Consortium and consisting – as the reader will enjoy immediately in the very next future – from two parts, which look so similar but in the reality appear to be so different.

In the entries of the first row of the table 3×9 *the consecutive numbers from 1 till 9 are written; in the entries of the second row the same integers but in some other order*

$a(1),\ a(2),\ a(3),\ a(4),\ a(5),\ a(6),\ a(7),\ a(8),\ a(9)$

are being presented. Finally in the third row in each column the difference of the corresponding numbers, which are written above them in the first and in the second row, subtracting the smaller number from the bigger number, is taken. If these numbers are equal then their difference is 0.

So that any such table looks like

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
$a(1)$	$a(2)$	$a(3)$	$a(4)$	$a(5)$	$a(6)$	$a(7)$	$a(8)$	$a(9)$
$\lvert 1-a(1)\rvert$	$\lvert 2-a(2)\rvert$	$\lvert 3-a(3)\rvert$	$\lvert 4-a(4)\rvert$	$\lvert 5-a(5)\rvert$	$\lvert 6-a(6)\rvert$	$\lvert 7-a(7)\rvert$	$\lvert 8-a(8)\rvert$	$\lvert 9-a(9)\rvert$

For the sake of simplicity let's take the "concrete" table:

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
4	**3**	**7**	**9**	**8**	**6**	**2**	**5**	**1**
3	**1**	**4**	**5**	**3**	**0**	**5**	**3**	**8**

We note immediately that numbers in each of the first and the second row are all different integers – it must be so according to the given conditions. Now when the numbers in the first and in the second row are given, the numbers in the third row being the absolute values of their difference in columns are of course also determined.

From the given table we see that although the numbers in the ***first*** *and in the* ***second*** *row* ***are different*** *the numbers in the* ***third*** *row* ***are not all different***

Now the reader is probably thinking from which side or under what formulation the new problem will occur?

The question or the new task for us will be formulated as follows:

Is it possible to rearrange the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 in the second row in such a way that corresponding differences in the third row would be all different?

In other words we are eager to avoid monotonicity in the third row trying to avoid the situation when we get the same difference in different entries of the third row. We note that if it was possible to enjoy such a situation with all these differences being different then all these differences would be 9 different integers from the interval [0; 8] or in other words, expressing the same, they would be exactly all integers 0, 1, 2, 3, 4, 5, 6, 7 and 8 in some order.

So we state that we are eager to find some reordering

$$a(1),\ a(2),\ a(3),\ a(4),\ a(5),\ a(6),\ a(7), a(8),\ a(9)$$

of the numbers

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9$$

such that the absolute values of their corresponding differences would be all different or would be the integers

0, 1, 2, 3, 4, 5, 6, 7, 8

taken in some order.

In a somehow paradoxical way that is possible and not very difficult to achieve:

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
8	**7**	**4**	**6**	**5**	**9**	**3**	**2**	**1**
7	**5**	**1**	**2**	**0**	**3**	**4**	**6**	**8**

And now when we are in so joyful psychological state of being and mind (no wonder, we have just achieved something), the following so similar question follows:

Is it possible to prolong or extend the table that we just constructed by adding one column or to do the same what was just being done in the case *N* = 9 also in "neighbouring case" *N* = 10:

Write all the consecutive integers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 in the first row, some rearrangement of them in the second row and get all different numbers as the difference in corresponding columns in the third row.

It would so nicely correspond to the natural motto:

No repeated integers in any of these three rows

But in a somehow strange and astonishing, yet even almost mystical way it seems to be either difficult or impossible to fulfil – all attempts to construct similar table lead to nothing, so something what we have done so easily for 9 consecutive integers somehow can not be implemented for 10 consecutive integers.

Seeing such a remarkable difference between these 2 neighbouring cases any German speaking person might ask –

Wo liegt hier der Hund begraben? –

Where is our dog dug?

Let's regard that situation more carefully or let's try to pay now all our attention for every possible detail, which might explain that strange situation when the almost unnoticeable changes of circumstances brought essential changes or, otherwise, remarkable differences.

Firstly note that the sum of 9 first consecutive integers

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$$

is, of course, an odd integer.

If we will "slightly" change the situation by adding number 10, then the sum remains odd because

$$(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) + 10 = 45 + 10 = 55.$$

If we will add then the next consecutive number 11 then the sum would no more odd because

$$(1+2+3+4+5+6+7+8+9+10)+11=55+11=66$$

is an even number.

This demonstrates that sometimes small changes are making the situation different and sometimes not. It depends. This is psychologically highly astonishing but logically understandable.

Sometimes small changes are "the last straw that broke camel's back" and in other cases they aren't. It may also be last but one and camel's back remains unbroken.

Another example similar to these regarded cases is the "same"difference in arranging the tables in the "neighboring" cases $\boldsymbol{N = 5}$ and $\boldsymbol{N = 6}$.

In the first case everything is again very easy or we are able to find the required table at once, e.g.:

1	**2**	**3**	**4**	**5**
5	**2**	**4**	**1**	**3**
4	**0**	**1**	**3**	**2**

and again we state that we are not able to present the similar table in the case $\boldsymbol{N = 6}$.

It should be added that in the last case we could examine all possible cases "by hand" and in the initial case $\boldsymbol{N = 10}$ also easily, using the calculator; but that would be not so interesting because we want to explain the situation using some "principal" or "theoretical" considerations and not by "asking at once" the computer.

So once again: what circumstances are able to change our situation so essentially?

BREAKTHROUGH XXIX (LIII). SOME PSYCHOLOGICAL CONSIDERATIONS ACCOMPANIED BY OTHER SIMPLE AND NICE ONES, BUT STILL FOR EVERYONE WHO WISHES TO THINK ABOUT AN ACCESSIBLE ROMANIAN PROBLEM

From the psychological or human point of view there are no other things, which be much worse than the following conclusion: if I'm not able to do something, or solve the problem in, say, 5 minutes, than this isn't worth wasting my time and doing it.

At the first glimpse such a behaviour seems to be logical alone from the reason that it might really happen: that situation or proposed problem is too hard for me, so it would be better to lay it aside forever and at least for a while. It that case I'm intuitively saving myself from the further attempts feeling that I'm not able to realize what I was asked to.

This is not an easy feeling. To say or to state "I'm not able to do this or that"

might be and is rather painful for it touches my natural pride.

That's why we are always feeling and repeating that the real activities teach us to be modest. They bring us to sometimes rather painful understanding that there are things, which you are not able **to do at once or even in a week, it happens than even in a month or even during several years.**

But not in vain is told (John 8, 32):

***You will** know the **truth** and the **truth** shall **make you free**.*

Otherwise it also ought to be added that the deepness of peoples' mind is different and not the same. Abilities to achieve or realize something are also different. But all these circumstances cannot change that main idea that everyone is able to learn astonishingly much everywhere and in each area.

In other words, it is always better to go on proceeding the efforts in trying to do or to arrange something instead of complaining how bad the world already is or our neighbours will be in near future.

Not every person may learn to play basketball like Dirk Nowitzki but everyone is able to bring itself to the condition to be able one day to throw in 10 fouls of 10.

That is the case in problem solving too.

Only do not be afraid if it appears necessary to read the text of the given problem thrice in order to be able to *answer the question: Am I aware what that problem asks me to do? Have I some ideas how to perform?*

Afterwards if you still stay at that problem, for instance, when you're not yet able to get rid from it, then sometimes it is of great help to do anything in the given direction.

If you are still not able to achieve something then we would like to advice you to lay it aside for a while. Then with new energy and force you are able to return back without losing any time for getting acquainted with the problem and details of it.

All that may take and takes a lot of human and psychological resources.

This isn't easy but it strengthens my mind and will, inspires positive changes in our personality and character and increases our self-confidence.

These are exactly the qualities that we sometimes need so desperately in some events of our life.

These are the most precious qualities and advantages which the exact sciences, beside of concrete knowledge, give to everyone who is dealing with them and who is involved in them.

You will enjoy similar influence doing every kind of activities with real content.

You should never start thinking without serious reasons that something is too difficult for you or that you are not able to carry it through. You shouldn't think that your efforts give you nothing, lead you nowhere and that in general all that industry is not worth your efforts and attention.

Again we would like to repeat that dealing with the problem and reading the formulation you never know what kind of problem you've met: standard exercise, challenging problem or the problem, which mankind still isn't able to solve.

But in every case the attempts to achieve something always pay back and are always useful: at least you'd get acquainted with the realities or repeating that what was being already told before:

You will (now) know (more about) the **truth** and the **truth** shall **make you (**more**) free** (than you've been).

Not in vain we are constantly repeating that Rome wasn't built in a day.

Take it easy when you are dealing with the real problems. Take your time. You will win or at least you will learn where the truth is.

As a kind of illustration we start regarding that proposed Romanian problem with a clear condition and more then understandable aim. Spending some efforts you'll always be able to achieve some progress or move forward.

This problem was proposed for Grade 7 and is taken from the Romanian Olympiad A. D. 2003

Again we would highly recommend the book [7]. Our problem is formulated on page 24 and is followed by perfectly simple and efficient solution.

By the way, Romanian Olympiad books about mathematics and wisdom are especially precious and valuable also by presenting some concrete, nice and in the same time realizable problems especially for these who just start to learn something about some equally nice and accessible real tasks.

You can at once enjoy the shortness and beauty of the formulation:

In how many ways is it possible to split the set of natural integers in two disjoint parts so that the sum of the numbers in one part is equal to the product of the numbers in another part?

So we are supposed to split the set

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9,10\}$$

in two disjoint subsets *A* and *M so that adding all the elements in A and multiplying all the elements in M we would get the same result.*

First intuitive feeling of heuristic or common sense naturally whispers to us that if this is possible to do then the set of the numbers which we are going to add, or the set *A*, must contain "much more" elements than the set *M* of the numbers which we are going to multiply.

It's based on remark that the product of natural numbers, generally speaking, exceeds their sum even if these numbers are as small as they are in our case – with the exceptions of the type "two times one" or "two times two".

Starting the solution we observe that the sum of all these first 10 consecutive natural integers or the sum of all possible members of M is

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$$

while already the product of the 5 smallest possible members of the set M is

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120.$$

It follows that the set M (or the set the elements of which we are going to multiply) can contain at most 4 elements.

We will classify all the possible cases with respect to the number of elements of the set *M* – the set the elements of which we are going to multiply.

Simply and prosaically we list all the possibilities:

(α) Set M contains exactly one integer;

(β) Set M contains exactly two integers;

(γ) Set M contains exactly three integers;

(δ) Set M contains exactly four integers.

We start consequently regarding these cases one by one.

The **case (α)** or the case when the set *M* contains exactly one integer is exhausted at once because it's impossible: imagine the set *M* consisting of one number, then the product of it (that integer) is at most 9 and the sum of all other elements or these of *A* is at least

$$55 - 9 = 46.$$

The **case (β)** or case when the set *M* contains exactly two integers will take more time for consideration. For *M* containing two integers we'll provide them the names – *x* for the smaller and *y* for the bigger number. Then their product xy ought to be equal to the sum of the other integers $55 - x - y$. Then

$$55 - x - y = xy$$

or

$$xy + x + y + 1 = 56$$

which is the same as

$$(x + 1)(y + 1) = 56.$$

In an abstract situation *x* and *y* being integers we ought to regard all possibilities of writing down 56 as a product of 2 integers. They are

$$56 = (-56) \cdot (-1) = (-28) \cdot (-2) = (-14) \cdot (-4) = (-8) \cdot (-7) =$$
$$= (-7) \cdot (-8) = (-4) \cdot (-14) = (-2) \cdot (-28) = (-1) \cdot (-56) =$$
$$= 1 \cdot 56 = 2 \cdot 28 = 4 \cdot 14 = 7 \cdot 8 = 8 \cdot 7 = 14 \cdot 4 =$$
$$= 28 \cdot 2 = 56 \cdot 1.$$

In our case x and y being positive integers $x + 1$ and $y + 1$ are greater than 1 so the first 9 cases (from the possible 16) are impossible.

Further on $x < y$ obviously implies $x + 1 < y + 1$, so the last four cases are impossible too.

It remains to regard the following possibilities

$$(x + 1)(y + 1) = 56 = 2 \cdot 28 = 4 \cdot 14 = 7 \cdot 8$$

Now because of *x* and *y* being digits the first and the second cases are impossible as well. Then from all 16 abstract possibilities to split 56 into product of two integer multipliers $x + 1$ and $y + 1$ with *x* and *y* being digits only one remains:

$x + 1 = 7$ and $y + 1 = 8$, giving the only solution

$$(x, y) = (6, 7).$$

So in the **case (β)** $M = \{6, 7\}$ and $A = \{1, 2, 3, 4, 5, 8, 9, 10\}$; indeed

$$6 \cdot 7 = 42 = 1 + 2 + 3 + 4 + 5 + 8 + 9 + 10.$$

In the following **case (γ)** set *M* contains already three numbers. Then again it is natural to arrange them in the increasing order before giving them names *x, y* and *z*:

$$x < y < z.$$

Then the required equality implies

$$xyz = 55 - x - y - z$$

or

$$xyz + x + y + z = 55.$$

Now we are going patiently to regard the cases $x = 1$, $x = 2$ and so on till it will be possible. We may note: if $x \geq 3$ then the last equality is impossible because then *xyz* would be at least $3 \cdot 4 \cdot 5 = 60$ indicating that the last equality doesn't take place. So there will be only two cases: **(γ1)** $x = 1$ and **(γ2)** $x = 2$.

Case (γ1):

Plunging $x = 1$ into the equation we get

$$yz + y + z = 54.$$

Acting as earlier we get

$$yz + y + z + 1 = (y + 1)(z + 1) = 55$$

and again the only possibility for y, z is to have

$$y + 1 = 5,\ z + 1 = 11 \text{ giving } y = 4,\ z = 10$$

or

$$M = \{x, y, z\} = \{1, 4, 10\}$$

(Indeed $1 \cdot 4 \cdot 10 = 40 = 2 + 3 + 5 + 6 + 7 + 8 + 9$.)

Case (γ2):

Plunging $x = 2$ into the equation we get

$$2yz + y + z = 53.$$

This allows us after the doubling of equation or writing

$$4yz + 2y + 2z = 106$$

and after adding 1 to both sides to split the expression into product writing

$$4yz + 2y + 2z + 1 = (2y + 1) \cdot (2z + 1) = 107$$

Now 107 being clearly prime we don't get any solution.

Now there remains the last case (δ) when M contains four integers x, y, z, t arranged in the usual increasing order

$$x < y < z < t.$$

We have

$$xyzt + x + y + z + t = 55.$$

Again only the case $x = 1$ is possible; otherwise $x \geq 2, y \geq 3, z \geq 4$ and $t \geq 5$, so $xyzt \geq 2 \cdot 3 \cdot 4 \cdot 5 = 120$ which shows that the latter equality is impossible.

So $x = 1$ is the only possibility to proceed in the **case (δ)**: we get

$$yzt + y + z + t = 54.$$

We'll sort now the subcases with respect to y, and so now only

$$y = 2$$

is possible; otherwise

$y \geq 3$, $z \geq 4$ and $t \geq 5$, giving $yzt \geq 3 \cdot 4 \cdot 5 \geq 60$ which never takes place in the last equality.

So now plunging $y = 2$ into the equality we get

$$2zt + 2 + z + t = 54 \text{ or } 2zt + z + t = 52.$$

Again after doubling and splitting we get

$$(2z + 1) \cdot (2t + 1) = 105$$

Remembering that z and t are digits and $z < t$ we see that the only possibility for $2z + 1$ and $2t + 1$ is the following one:

$$2z + 1 = 7 \text{ and } 2t + 1 = 15$$

leading to $z = 3$ and $t = 7$ and giving $(x, y, z, t) = (1, 2, 3, 7)$. This is indeed a good solution because in this case

$$M = \{1, 2, 3, 7\} \text{ and } A = \{4, 5, 6, 8, 9, 10\} \text{ and}$$

$$1 \cdot 2 \cdot 3 \cdot 7 = 42 = 4 + 5 + 6 + 8 + 9 + 10.$$

The answer: *We have the following splittings of the set of first 10 natural integers into 2 disjoint subsets with the product of integers in one set being equal to the sum of remaining integers in another set:*

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} = \{6, 7\} \cup \{1, 2, 3, 4, 5, 8, 9, 10\}$$

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} = \{1, 4, 10\} \cup \{2, 3, 5, 6, 7, 8, 9\}$$

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} = \{1, 2, 3, 7\} \cup \{4, 5, 6, 8, 9, 10\}.$$

BREAKTHROUGH XXX (LIV).
THE PROBLEM OF ALBANIAN GRAND-DAD

This problem we have firstly seen in the Albanian problem and wisdom sheets and this naturally explains also the title of the chapter.

It is very well known that in our countries the families don't provide as many children any more as before. Still after reading the text of the problem it might be also understood that not everywhere on our globe and even in Europe the things are as bad as they seem to be somewhere else.

So let's get acquainted with that interesting and demographically highly optimistic problem.

To some country school 20 children are attending. The school is extremely friendly and internal – that's no wonder knowing that any two of these 20 have a common Grand-dad.

We are asked and invited to establish whether in each such school there must exist a Grand-dad having in that school at least 14 grandchildren?

We couldn't imagine a person on the globe not knowing that every person has exactly two grand-dads: one is the father of his father while the other is the father of his mother.

With that trivial remark, which we've done for the sake of simplicity in order to show that we understand the nature of things, we start with our exciting investigations.

What's now? Is it really possible to prove it?

Is there an easy way to do it?

Are there (sometimes) easy roads to the difficult tasks?

Firstly we would like to mention that such a situation in general is possible. There might be a person having 20 and even more – but still finitely many – grandchildren in some school: for instance, a person might happen to have 20 children and each of these 20 children might have his own child exactly in that class.

But this is a so-called separate case.

How could we proceed in the general case?

We are expected to formulate the answer for all possible cases.

That means to provide the "general" proof for all such schools or to indicate some particular school where every 2 from these 20 do have the common Grand-dad but where there is no Grand-dad with at least 14 grandchildren in that school.

One of most successful treatments as in the life so in the problem solving is the illustration or refreshment of the presented content and its data or of any other connected material.

In order to achieve some progress let us imagine that we are also attending to that school. We arrived just on that day when all children were obliged, that is, kindly but resolutely asked, to bring the separate photos of their Grand-dads. That means, 2 photos each. Their young but already wise teacher asked them to raise both hands with a photo of a Grand-dad in each.

So we see 20 children, 40 raised hands and also 40 photos in these hands. Of course, perhaps some persons are to be seen only on one picture or photo, but probably quite a lot of them are on several photos – remember that each two kids have the same grand-dad.

*Now we declare some help for that teacher and are helping him to gather these photos and to count the number of **different** persons presented on these photos. The kids are still holding their hands raised high with the photos – this happened to be their first school day after vacations, that's why they are unreally patient.*

*And now, before counting the number of **different** persons presented on these photos, the teacher asks the question which at the first glimpse appears to be extremely strange.*

He asks whether

(*A*): 3 or less persons

or

(*B*): more than 3 persons

are presented on these 40 photos?

Why 3 is so interesting for our investigations?

What a difference does it make?

Nevertheless we'll try to answer the question.

Case (*A*) with at most 3 persons seen on all these 40 photos.

Now because

$$40 : 3 > 13$$

it follows that there is a person which is presented on more than 13 photos.

If I am seen on more than 13 photos that clearly indicates that I'm present on at least 14 photos.

So in the case *A* the existence of a person with 14 grandchildren in that school is already granted and so **the case (*A*) is exhausted.**

Now what remains to do is to regard the

Case (*B*), which means that there are more than 3, that is, at least 4 different persons on these 40 photos.

Now we will apply the following illustration of data: we'll give the names D1, D2, D3 and D4 to these first 4 ***different*** *grand-dads located on the photos.*

Then the key situation will be as presented in the table below:

First child	**Second child**	**Third child**
Holding photos of D1 and D2	**Holding photo of D3**	**Holding photo of D4**

Now what about the second photo in the other hand of the second child? Please remember that the second child ought to have the common grand-dad with the first child.

But the first child is holding photos of D1 and D2 so that the second grand-dad on the other photo of the second child must be either D1 or D2. No other possibility is left. The conditions must hold.

Without the loss of generality we may assume that the second child is holding the photo of D1.

Then we have the following picture:

First child	**Second child**	**Third child**
Holding photos of D1 and D2	**Holding photos of D1 and D3**	**Holding photo of D4**

And now the following serious question: what is the second grand-dad on the second photo of the third child? Remember that he has a common grand-dad with the first as well as with the second child.

If the third child holds the photo of D1 then he clearly has a common grand-dad with both of them.

And what if not? What if he doesn't hold the photo of D1?

Then having the common grand-dad with the first child and not holding the photo of D1 the third child is forced to hold the photo of D2. So the third child is holding photos of D2 and D4. Let's again take a look to the table

First child	Second child	Third child
Holding photos of **D1 and D2**	Holding photos of **D1 and D3**	Holding photos of **D2 and D4**

But that situation is impossible because of then the second and the third child have no dad in common – grand-dads D1, D2, D3 and D4 are all different!

That means the third child must also hold the photo of grand-dad D1, so we have

First child	Second child	Third child
Holding photos of **D1 and D2**	Holding photos of **D1 and D3**	Holding photos of **D1 and D4**

Now consider any other fourth child and ask:

First child	Second child	Third child	(Any) fourth child
D1 and D2	D1 and D3	D1 and D4	What does he hold?

If he holds the photo of D1 then everything is settled and proved.

If he doesn't hold the photo of D1 then he is forced to hold the photo of D2 because he has a grand-dad in common with the first child. But then from the very same reason he is forced also to hold the photo of D2 as well as the photo of D3.

Stop, that's impossible because even Alice in Wonderland couldn't imagine the child possessing 3 different grand-dads.

It follows then that if there are 4 or more different grand-dads on these 40 photos then they all have a common grand-dad.

Remember that if there are at most 3 dads on 40 photos then from the simplest properties of mean value or of the common sense we have that there is a grand-dad which is located on more then 40/3 photos, that is, there is such a grand-dad which is located on at least 14 photos.

So we've settled the question for all cases and repeatedly state:

If every 2 from any 20 person do have a common grand-dad then there are at least some 14 of them which all are grandchildren of the same grand-dad.

BREAKTHROUGH XXXI (LV). ONE SUDOKU AND TWO WORDS ABOUT

Sudoku is a source of infinite optimism for each who tends to think that nowadays people are not eager to apply their minds in order to find the proper way by obeying strict but understandable rules.

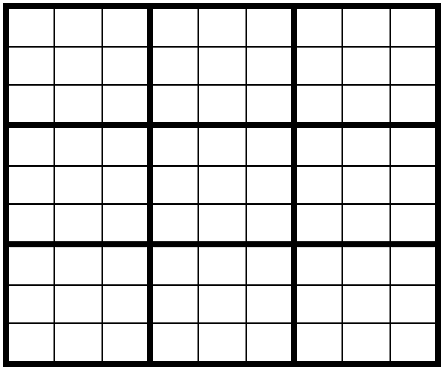
Also it would be difficult to find a person, which hadn't solved some of them.

We cite these rules rather for the sake of completeness.

As it is known, solving a sudoku puzzle can be very tricky because the rules of the game are as simple as possible.

It wonders the author deeply that this game hasn't been invented, say, in ancient Greece. Some explanation could be, e.g., lack of paper and probably ink.

But why it wasn't invented, say, in the Middle Ages, after Gutenberg's invention of printing, would be difficult or close to impossible to explain. Think about these world-wide known magic squares presented in the picture of Cranach. These are of much more complicated nature.



Sudoku affairs usually are taking place in a grid of nine by nine squares as shown in the picture above. The 9×9 square is in the most natural way divided into nine 3×3 squares as it is also indicated in the same picture.

Further on in each horizontal row, as well as in each vertical column and also in each 3×3 subsquare each of numbers

$$1, 2, 3, 4, 5, 6, 7, 8, 9$$

might and must appear exactly once.

In each sudoku puzzle some digits are already given and these digits are in no way to be changed or replaced. The puzzlers' job and honorable duty is to fill the remainder with digits respecting the rule that in each horizontal row, in each vertical column and in each of the nine 3×3 subsquares each number

$$1, 2, 3, 4, 5, 6, 7, 8, 9$$

appears exactly once.

A good sudoku puzzle has always only one solution.

In all suitable instructions it is always told and repeated that

SOLVING A SUDOKU PUZZLE DOES NOT REQUIRE KNOWLEDGE OF MATHEMATICS; SIMPLE LOGIC SUFFICES. (INSTEAD OF DIGITS, OTHER SYMBOLS CAN BE USED, E.G., LETTERS, AS LONG AS THERE ARE NINE DIFFERENT SYMBOLS.)

The first part of the cited sentence "**SOLVING SUDOKU PUZZLE DOES NOT REQUIRE KNOWLEDGE OF MATHEMATICS"** is the worst sentence I've read in my life.

It indicates two things:

1. In the consciousness of many people the word "mathematics" isn't bound with the most pleasant events of our school and life history.

2. It is better, speaking about sudoku and using the right words **"DOES NOT REQUIRE… MATHEMATICS",** to separate that prospering sudoku movement from mathematics.

This is understandable but not quite right and not essentially true.

It only could be cited again:

But not in vain is told (John 8, 32):

You will *know the* ***truth*** *and the* ***truth*** *shall* ***make you free****.*

The following historical remark doesn't also imply any sadness at all.

In spite of the game's apparent simplicity, sudoku can be *highly* addictive. While the first sudoku puzzle was published as early as in 1979 (back then, it was called "Number Place"), the game's popularity flourished in 2005; it can now be found in many newspapers and magazines far and wide all over the world.

We would like to present to you the sudoku with the smallest known number of initially given entries.

There are only 17 of them. This is in average less than 2 in each row, each column and each 3 × 3 subsquare.

	8		**2**		**6**			
1	**6**							
6	**1**				**4**			
	9					**3**		
						7		**5**
							1	
				7			**8**	
		7		**3**				

We would be glad if the reader would find sudoku with lesser number of initially given digits but still with only one possible solution.

For details vide [8].

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