The LAIMA series

Romualdas KAŠUBA

ONCE UPON A TIME I SAW A PUZZLE

PART I

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R. Kašuba. Once upon a time I saw a puzzle. Part I.

Rīga: University of Latvia, 2008. – 59 pp.

The book demonstrates psychological aspects of problem solving on the basis of contest problems for junior students. Nevertheless, the approaches discussed are of value also for highest grades, for teachers, problem composers etc. The text can be used by all those who are preparing to research in mathematics and/ or to math contests.

The final version was prepared by Ms. Dace Bonka.

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"Let us take them in order.
The first is the taste…"

The Hunting of the Snark,
by Lewis CARROLL, Fit the Second

INSTEAD OF INTRODUCTION

Once upon a time there happened a day when the Lithuanian team-contest in mathematics was born. Speaking simply and prosaically, some mathematical event came into being. In that team-contest, five students, usually from the highest school grades, during 4 hours have to deal with 20 problems (you may notice that 5 times 4 is also 20 and admit it to be a remarkable fact).

That 1st version of that team-contest happened A.D.1986 and since then it is repeated year by year. With the time following problem occurred. Imagine that you are going to Vilnius, the capital city of Lithuania, to participate in that team-contest and I am your younger brother. I also would like to go to Vilnius with you. And the elder brother answers, that he has nothing against it. But in order to go to any capital or otherwise remarkable city it is better to possess some reason or pretext.

So the author having heard about that and wishing to help in all such and similar cases invented some pretext for involved younger brother to go to the capital city with *the elder brothers. He organized, or invented, the associated individual contest for younger brothers or for the younger grades.*

The organizer had also his own daughter, approximately of that age, only a bit younger, so his understanding on the willingness was even deeper.

That first individual contest for youngsters happened A. D. 1999 and the problems were proposed for younger sisters and brothers of grades 5, 6 and 7.

Two years later it was split into two subsections with different problems proposed for grades 5 and even 6 and another for the forms 7 and even 8.

This year already the 10th edition of that contest took place.

The author of these lines got a very honourable proposal to prepare the English version of these problems together with solutions.

The original intention was to include all of them in one volume. But rather because of some technical difficulties or otherwise because of shortage of time it was decided firstly to prepare the first 3 Olympiads together with some solutions.

And one more thing should necessarily be told and explained and even as well as possible. That is the adoption or harmonization. Otherwise, introducing *characters to make the solution itself a part of their achievements.*

A good part of the first Olympiad problems was formulated "without heroes". Then in order to make all that more attractive all the heroes occurred later, sometimes even against the will of the composer.

It must be told in a very clear way that practically all problems as such are taken from other sources and only after that they are adopted, reformulated or otherwise structured inventing some steps into which the question of the problem is divided also in order to make all that more attractive.

These attempts by the author were welcomed by quite a lot of involved persons: by students, teachers and colleagues. Taking that into account and following also some other reasons and advices I reformulated also all problems of the previous years.

Listening to all that it is understandable that I tried to present also the solutions as some kind of discussion between the persons involved and some imaginary Advisory Board.

The readers may have their judgment whether the author succeeded in achieving his goals. All remarks especially those critical ones would be extremely welcomed.

The author is thankful for the noble editors of the LAIMA series for constant inspiring of the author to do something.

I am very much indebted to the Father of LAIMA project Professor Benedikt JOHANNESSON, who is the constructive optimist from any serious point of view you only may invent or imagine.

I am also very much indebted to Professor Agnis ANDŽANS, who was the first reviewer of my first Lithuanian book. He inspired me to make also the translation of that me first book into English.

I am also very much indebted to Aivaras NOVIKAS for his eagerness to discuss all things with me every day. His constant linguistic advices were of great help and importance.

And of course also great are my thanks to Mrs. Dace BONKA, who has prepared already two of my English manuscripts and, as I hope, will have enough patience to prepare also the third one.

Romualdas Kašuba

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ABOUT LAIMA SERIES

In 1990 international team competition “Baltic Way” was organized for the first time. The competition gained its name from the mass action in August, 1989, when over a million of people stood hand by hand along the road Tallin - Riga - Vilnius, demonstrating their will for freedom.

Today “Baltic Way” has all the countries around the Baltic Sea (and also Iceland) as its participants. Inviting Iceland is a special case remembering that it was the first country all over the world, which officially recognized the independence of Lithuania, Latvia and Estonia in 1991.

The “Baltic Way” competition has given rise also to other mathematical activities. One of them is project LAIMA (Latvian - Icelandic Mathematics project). Its aim is to publish a series of books covering all essential topics in the area of mathematical competitions.

Mathematical olympiads today have become an important and essential part of education system. In some sense they provide high standards for teaching mathematics on advanced level. Many outstanding scientists are involved in problem composing for competitions. Therefore “olympiad curricula”, considered all over the world, is a good reflection of important mathematical ideas at elementary level.

At our opinion there are relatively few basic ideas and relatively few important topics which cover almost all what international mathematical community has recognized as worth to be included regularly in the search and promoting of young talents. This (clearly subjective) opinion is reflected in the list of teaching aids which are to be prepared within LAIMA project.

Twenty five books have been published so far in Latvian. They are also available electronically at the web - page of Correspondence Mathematics School of the University of Latvia http://nms.lu.lv. As LAIMA is rather a process than a project there is no idea of final date; many of already published teaching aids are second and third versions and will be extended regularly.

Benedikt Johannesson, the President of Icelandic Society of mathematics, inspired LAIMA project in 1996. Being the co-author of many LAIMA publications, he was also the main sponsor of the project for many years.

This book is the 7th LAIMA publication in English and the 32nd in general.

THE FIRST LITHUANIAN MATHEMATICAL OLIMPIAD FOR YOUNGSTERS (1999)

Grades 5, 6 and even 7

1. FINDINGS AND ENCODINGS OF JOHNNY DEAR

Johnny-Dear dreams that he can encode the following multiplication. Here as usual in both multiplicands different digits are naturally replaced by different letters and equal digits – by identical letters. All other digits are replaced by the same sign x.

Is it possible for Johnny-Dear, knowing so little, to decode that multiplication till the very last unknown digit?

$$
\begin{array}{r}
FIND \\
\underline{FIND} \\
xx\,x\,x \\
xxxx \\
\underline{xxxx} \\
xxxxxx\,x\,x
\end{array}
$$

2. SPLIT THAT RECTANGLE IN TWO AND MAKE THE SQUARE

Smarty Anne would be very eager firstly to split the rectangle 4×9 into two parts and then form the usual square from these two parts.

Do you believe that Smarty Anne is indeed able to succeed in doing that?

3. WHO'S FROM THE LIST AND WHO'S NUMBER ONE HERE?

The bravest captain Robinson Crusoe sailing in the billowy ocean of Numbers hopes to reach the Island where all numbers having exactly 100 digits have their place of residence. Approaching that Island the all-bravest captain has two wishes:

(☺) to meet any 100-digital number divisible by 100 and with the sum of digits also 100;

(☼) assuming the support of all who might feel concerned Robinson wishes to find the smallest of all such numbers.

4. (MEGA)PLATE OF CHOCOLATE AND HOW TO DIVIDE IT?

Karlsson and Little Brother found the mega chocolate plate of size 1999×1999 and started in turn to eat it obeying the following rules. Karlsson, who starts, may in the usual way take out from that (mega)plate any piece of size 2×2 and Little Brother – only those of the smallest 1×1 size. If the player is not able to perform any more then the other player takes the remaining part of chocolate.

Is it possible to anyone of them to perform in such a way that he would always get more than the half of the whole plate of chocolate independently of what the other player is doing?

SOLUTIONS

A.D. 1999, Grades 5, 6 and even 7, problem 1. Johnny-Dear with his encoding

Of course, the long multiplication is the most powerful source of information.

We may and we even must ask: why is it so? What are the reasons for that?

These are questions, which may not necessarily occur to our Johnny-Dear who has the aim first of all not to contemplate but rather to find the right answer. But the question of the kind will obviously remains also for the Johnny-Dear.

If only he will be able to master that.

This is so powerful first of all because the “long multiplication” preserves and presents all intermediate structural information and afterwards, when some essential part of it is hidden or lost – what appears sometimes to be the same – even then we are able from that what is remaining – sometimes from rather small part – to recover all the information we need or are asked for.

With that somewhat classical remark all we are able to state after looking to what is presented is that we are calculating the square of some 4-digital number, all the digits of which are different, knowing not more than how many digits all these intermediate results and final result possess.

We know nothing more and also Johnny-Dear would know nothing more from the first sight as well as in general.

And is it really so, that nothing more is needed in order to recover all information?

Let us see and enjoy Johnny’s solution.

Let us imagine that we have some approach also to his thoughts.

Examining with the quick glimpse what is being presented within

$$\begin{array}{cccccccc}
 & & & & F & I & N & D \\
 & & & & F & I & N & D \\
\cline{5-8}
 & & & & x & x & x & x \\
 & & x & x & x & x & & \\
 & x & x & x & x & & & \\
\hline
x & x & x & x & x & x & x & x
\end{array}$$

he immediately concluded that

$$N = 0$$

because in the “computational part” there are only three shifts instead of the possible usual four and that exactly the second computational shift from above is shifted not by one digit to the left as it is usually done but just by two.

Technological reason for that is that it is allowed (such is the common convention) to omit intermediately shifts consisting of all zeroes.

That was possibly the first of Johnny’s conclusions.

The second conclusion of his might be that he has interest in number of digits of the final result and how it is located with respect to intermediate shifts and especially in our case to that lowest one. From that location of the final result Johnny concludes at once that the first digit of the final result must be necessarily 1 and so the first digit in the lowest intermediate shift is necessarily 9. Plunging that information he may get and gets

$$\begin{array}{r}
F\,I\,0\,D\\
\underline{F\,I\,0\,D}\\
x\,x\;0\;x\\
x\,x\,0\,x\\
\underline{9\,x\,0\,x}\\
1\,x\,x\,x\,x\,x\;x\;x
\end{array}$$

Taking the second glimpse he may have the following insights or, simply speaking, may notice the further details like these we are also going to mention:

(A) The second digit of the end result – that is also as clear as the day – that second digit must be 0;

(B) The “9” in the lowest shift of the “middle block” indicates that the first digit of his squared number must be 3 (this is the only possibility to get 9 in the lowest intermediate shift); so

$$F = 3.$$

(C) The squared number being 4-digital as well as also all intermediate results being 4-digital allow us conclude together with Johnny-Dear that all remaining digits of that squared number also do no not exceed 3 (otherwise some intermediate shift would contain more than 4 digits).

$$\begin{array}{r}
3\,I\,0\,D\\
\underline{3\,I\,0\,D}\\
x\,x\;0\;x\\
x\,x\,0\,x\\
\underline{9\,x\,0\,x}\\
1\,0\,x\,x\,x\;x\,x\;x
\end{array}$$

The final conclusions – as well as those of Johnny’s or even ours – might contain the following statements or a small observation that for I and D being different and smaller than 3 there are only the following 2 possibilities left (recall that digit 0 is already used):

$$I \text{ is } 2 \text{ and } D \text{ is } 1$$

or, vice versa,

$$I \text{ is } 1 \text{ and } D \text{ is } 2.$$

The second case is impossible because then we would get the number 3102 and that number appears too small because the squared number

$$3102 \cdot 3102 = 9622404,$$

containing only 8 digits instead of needed 9, would be “too small”.

So it remains only

$$I = 2$$

and

$$D = 1$$

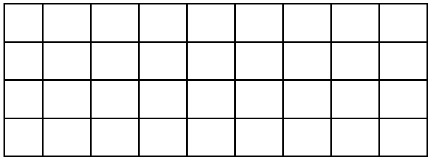
leading us with Johnny-Dear to the right answer 3201, because

$$\begin{array}{cccccccc}
 & & & & 3 & 2 & 0 & 1 \\
 & & & & 3 & 2 & 0 & 1 \\
\cline{4-8}
 & & & & 3 & 2 & 0 & 1 \\
 & & 6 & 4 & 0 & 2 & & \\
 & 9 & 6 & 0 & 3 & & & \\
\hline
1 & 0 & 2 & 4 & 6 & 4 & 0 & 1
\end{array}$$

The answer.
***FIND* = 3201.**

A.D. 1999, Grades 5, 6 and even 7, problem 2. Smarty Anne’s square and how did it all happen

What may Smarty Anne imagine and have in her mind when she starts? Many things, of course. But it appears somehow natural to suggest that in one form or another some representation of that rectangle of the size 4×9



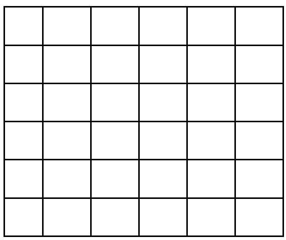
will appear in front of her mind's eyes.

The second possible impression seeing something like expressions 4×9 is to multiply participating numbers, to get 36 and then to ask whether it has some connection to original shape. Naturally it has, because it is the area of the whole rectangle.

And what about the relations of that area to the area of the square Smarty Anne is looking for?

Of course, since the area remains the same so the area of the square is also 36.

If the area of the square is 36, we set that the side of the square is exactly 6. But if the side of the square is 6, the square itself looks like this:



So the task is to get this from the initial rectangle.

X								
								Y

Somehow we might imagine that the topmost left corner of the rectangle 4×9 becomes or remains also the topmost left corner of the square we are looking for, both being located in the first part of the section we are going to make, and the opposite remote bottommost right corners both are in the other part. Let us denote these corners in the projected first parts by X and those corners in the other projected parts as Y. This is similar as if we were providing some guidelines for the sake of better orientation. Or we even might say that we are introducing the frame of references or even that we are dealing with some specific coordinate system.

So looks this with the rectangle we are starting with, and the similar picture with the square we are planning to get would be the following:

X					
					Y

If we are planning to get the square with one cut from rectangle so we might assume that some row or column containing X in the corner might become the first row or the column of the square we are trying to rearrange. But we are expected to make only one cut. So we state that the column of the rectangle containing X ought to contain 6 entries in order to become the first column of the square but it doesn't – it contains only 4. (Similarly about rows.) It means that we will try to take the row of the rectangle containing X and mark there the first row of projected square – it will be enough because it contains 9 entries and we will take the necessary 6 marking them all with X at once. So after first phase of marking we have in our rectangle

X	X	X	X	X	X			
								Y

Dealing symmetrically in the opposite corner with the entries marked by Y which will be located in another piece we would get

X	X	X	X	X	X			
			Y	Y	Y	Y	Y	Y

If we will make the similar marking for the second line once again the similar portion

X	X	X	X	X	X

And marking the corresponding one for Y's we would get some two shifts in one corner with X's and also some two shifts with Y’s on the opposite corner, which will be looking as follows

X	X	X	X	X	X			
X	X	X	X	X	X			
			Y	Y	Y	Y	Y	Y
			Y	Y	Y	Y	Y	Y

But then the region with X has common boundary with Y indicating already some part of the possible cut. There are remaining still uninvolved two regions of the size 2×3 which we for the short while may denote by bold **Z** so indicating them additionally in the picture:

X	X	X	X	X	X	**Z**	**Z**	**Z**
X	X	X	X	X	X	**Z**	**Z**	**Z**
Z	**Z**	**Z**	Y	Y	Y	Y	Y	Y
Z	**Z**	**Z**	Y	Y	Y	Y	Y	Y

But from two 3×2 parts we may arrange the remaining two rows of the square attaching the lower part with **Z’**s

Z	**Z**	**Z**
Z	**Z**	**Z**

to the topmost left part with X making the X region after renaming of Z’s to X’s to look as follows

X	X	X	X	X	X
X	X	X	X	X	X
X	X	X			
X	X	X			

making from the rest part of the 4×9 square the remaining part of the section – that with Y's.

We will indicate all the X and Y entries in the picture below. The boundary between X and Y became the border and will be the boundary line of these two cuts.

X	**X**	**X**	**X**	**X**	**X**	Y	Y	Y
X	**X**	**X**	**X**	**X**	**X**	Y	Y	Y
X	**X**	**X**	Y	Y	Y	Y	Y	Y
X	**X**	**X**	Y	Y	Y	Y	Y	Y

Finally we present the square we get :

X	**X**	**X**	**X**	**X**	**X**
X	**X**	**X**	**X**	**X**	**X**
X	**X**	**X**	Y	Y	Y
X	**X**	**X**	Y	Y	Y
Y	Y	Y	Y	Y	Y
Y	Y	Y	Y	Y	Y

The answer.

Smarty Anne can succeed – as it is indicated in the pictures above.

A.D. 1999, Grades 5, 6 and even 7, problem 3.
The numbers of the brave captain McDigit and even some smaller numbers with the similar flavor.
Simplification of the expected solution

Let us for a joke reduce the problem to some even simpler problem by taking away or removing one

zero from all places where we've met some two of them.

So in such a case the problem will become nicer because one zero is much more attracting than some two of them.

The grandson Matthew of the bravest captain of the billowy Ocean of numbers Mr McDigit dreams to reach the following island where all the numbers having exactly 10 digits have its place of residence. According to strong will of the all-bravest captain that 10-digital number, which he wishes to meet desirably, ought to be divisible by 10 and 10 ought to be also the sum of its digits.

The captain is also deeply interested in disclosing the smallest such number. Could you help him in the case of urgency or strong need?

Solution of the simplified problem.

The first thought in Matthew's head and mind when dealing with it probably was the following one:

It is enough for a number which is really intended to be divisible by 10 to have a 0 as its last digit – nothing more is needed – how simple are some matters in our complicated world!

So that number looks like

$$ABCDEFGHI0$$

With all these others A, B,…, being arbitrary one-digit numbers taking care only of the circumstance that

$$A + B + C + D + E + F + G + H + I = 10$$

or 9 numbers with the sum 10 is needed, so that, e. g.

$$1, 1, 1, 1, 1, 1, 1, 1, 2$$

would give us – as one of the possible answers – the number

1 111 111 120.

It remains to present the smallest possible such integer number – and the smallest possible integer number with any property always exists – we are always able to select the smallest possible number assuming that there exists at least one wanted number with that property.

Or simply speaking – if there is at least one such a positive integer, then among them there is also the smallest one.

Let us notice that we do not care about the nature of that problem – we do care only about the existence of one example – and such an example in our case is being presented above as a number

1 111 111 120.

The second thought of the grandson was the memory about the need for the answer to be 10-digital number – that means that its first digit must be at least 1 and another ones - as small as possible – but still first nine gather 10 as the sum of digits.

Then for example the integers

1 011 011 040, 1 000 020 070, 1 000 000 710

suit much better for the smallest such an number.

It is clear that we must start looking for the smallest 10-digital number as the right answer from the first digit 1, and then to try to carry the numbers making the digit sum which must be 9 as far as possible to the right. This is possible to achieve writing zeros after 1st digit, which is 1, taking care only about that the sum of remaining 8 integers would be 9

1 000 000 090.

Solution of the simplified problem is

1 000 000 090.

Now solution of the problem of the Captain McDigit is the imitation of what we just did.

1. First thought of Mr. Mc Digit is that the number divisible by 100 must end with 00.

2. Its first digit must be at least 1.

3. And then after this first 1 must be assisted by as many zeros as possible taking care only about that numbers preceeding the last two zeros must be as big as possible and with these digits preceeding the last two zeros we must gather the number sum

$$100 - 1 = 99$$

or taking

$$99 : 9 = 11$$

9's and so getting the number

1 000 …. 009 999 999 999 900,

which has first digit 1, then this 1 is followed by

$$100 - 1 - 11 - 2 = 1000 - 14 = 86$$

zeros, then these 86 0's are followed by 11 9's and then by the resting two 0's.

Solution of the original problem of the bravest captain McDigit is the 100-digital number

1 000 000 …. 009 999 999 999 900

86 zeros *11 nines*

A.D. 1999, Grades 5, 6 and even 7, problem 4.

Karlsson and Little Brother once were consuming in turn the mega chocolate plate of remarkable size 1999×1999, but for the sake of us all and, first of all, for the clearer solution cutting down and so remarkably reducing their appetites.

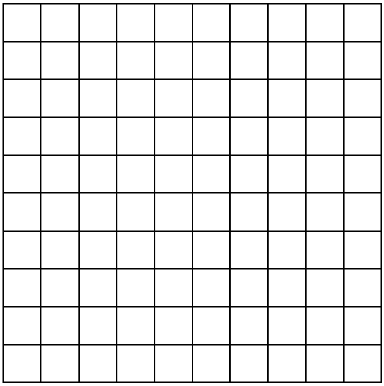
The essence of the problem is that Karlsson is allowed to take with each his move four times as much chocolate as Little Brother is – namely Karlsson is allowed to take in one move as 2×2 parts from every part of traditionally squared big and large chocolate while Little Brother in turn takes only 1×1 part.

So from traditional point of view Karlsson is in a privileged state with each of his moves taking four times as much as Little Brother does and the only possibility for Little Brother is the possibility always to make a move. Correspondingly, his strategy could be to remove from the chocolate each fourth 1×1 part "in a regular way".

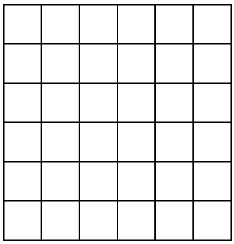
Let him try to realize it.

It is clear that the size of the chocolate has only the psychological meaning so we are always reducing it in order to milden the psychological pressure and then for the sake of simplifying let them eat or divide the 10 x 10 chocolate.

Such chocolates may be presented "up to the last its 1×1 piece" as follows:

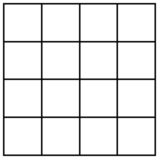


Imagining that the plate is still too big – then let’s take again the smaller part of it – the 6×6 chocolate plate:



If also now you have some doubt about the way in which Karlsson and Little Brother may act – so take the smaller part of even that – the 4×4 plate with the promise to make stop with our reducing: 4×4 is not so big and the things, which are going to take place there from the psychological point of view won’t look thrilling or complicated more.

Still we, together with Karlsson and Little Brother, do hope that they will remain involved:



So what to do and what might be undertaken?

Let us repeat that the problem would appear not as challenging as it might be if Karlsson would win. It’s no wonder and no attraction because he starts and with each his move swallows 4 times as big 2×2 pieces as the Little Brother does.

So the Little Brother hoping to get not less than a half of the whole chocolate must stop Karlsson after his second move because after the second move Karlsson would consume

$$4 + 4 = 8$$

from 16 smallest units of our plate – and this is already a half.

How could Karlsson be stopped by that young gentleman Little Brother? What a plan might be invented in order to realize that honorable entertainment? How to arrange that Karlsson makes once, takes again and “that was the end of sweet Molly Malone?”

We already promised that they go down in order to take even more smaller pieces and even keeping on that promise we might imagine that this 4×4 is in our mind fast divided, or better to say, consists of four 2×2 parts.

X X   X X
X X   X X

X X   X X
X X   X X

So these could be exactly 4 pieces for Karlsson if Little Brother would disappear or at least would be present during that whole process of consuming. But Little brother is present and in 2 seconds he is already equipped with the clear plan.

The essence of it is:

Karlsson is taking any piece he wishes – I let him start first – and then I will take something from any other so eliminating that piece from the "other takings".

And if there are still unspoiled pieces for Karlsson for the second taking so then I will spoil the last 2×2 piece taking some 1×1 from it.

How this could be brought to some perfection so that Little Brother could convince Karlsson even before the game?

Is this possible? Is anybody able to present such a clear plan before consuming?

Yes, it is!

So Little Brother demonstrated Karlsson the list with these 16 entries of the 4×4 square. In that square we expect once again to enjoy 16 entries, all of them being blank – and we have seen 16 entries indeed, but not all of them were blank, because some 4 of them were shaded black. We indicate these fields with the capital letter **X**:

	X		**X**
	X		**X**

And we have heard the two explanations which Little Brother made to his opponent. They were indeed as clear as a day and convincible.

Karlsson is free in his choice, but nevertheless with each of his takings:

1. Karlsson will remove exactly one entry with **X**.

2. And I, Little Brother, will remove any other of remaining entries, which now are marked with **X** and which are exactly my size or my tit-bit.

3. So Karlsson would be always stopped at latest after his second turn.

This indicates that the remaining part of chocolate will be given to Little Brother.

We remind that 2×2 part is already and exactly one piece or tit-bit for Karlsson.

And Karlsson nodded agreeing with everything.

It means that in the given case if both are as clever as possible – and they both are – each of them will get exactly the half – each of them 8 smallest 1×1 parts.

And in the general case exactly the same idea will work.

THE SECOND LITHUANIAN MATHEMATICAL OLYMPIAD FOR YOUNGSTERS (2000)

Grades 5, 6 and even 7

1. DRAGONS EMPTYING THE LAKE OF QUEEN VICTORIA

In the lake of Queen Victoria quells are spraying and some dragons are non-stop consuming the water of it.

If it were 183 of them then they would drink out the whole water capacities within one day.

If it were 37 of them then they would drink out the whole water capacities within 5 days.

In what time would the whole water of the lake be drunk out by the single dragon?

2. KINGS ARTHUR'S ATTENTION FOR DIVIDING THE PLANE AS A KINGDOM

The king Arthur wishes to indicate the following thing of importance: in what number of pieces may the plane be divided by the three lines lying in that plane?

3. EXERCISE FOR SIR LANCELOT

Sir Lancelot used every possibility to train his abilities in all possible ways. So once during a happy hour he felt a very strong wish and was very eager to calculate what is the smallest number, which he could get, taking up the sum of digits of multiples of 23.

4. PINOCCHIO AND MELVIN EXCHANGING THE SINGLE TILE

The bottom of the square box of the size 2000×2000 was completely covered by Pinocchio using tiles of the sizes 2×2 and 1×4. Afterwards Melvin had taken all of them away for some cleaning and then one of them, that of size 2×2, went to pieces.

Melvin, which is so excited and feels herself guilty, still believes that it would be possible to replace that 2×2 tile by another tile of size 1×4 so that the bottom of the box will be again completely covered by all these tiles while Pinocchio remains very sceptic about such a possibility.

Who of them – Melvin or Pinocchio – is right and why?

SOLUTIONS

A.D. 2000, Grades 5, 6 and even 7, problem 1. Drinking out the whole lakes with the uniformly functioning quells and equally drinking dragons

Of course, it will be again some system of equations. It's no wonder because of when several conditions like now are given then each condition after we'll give the name for the heroes would become an equation and we will be expected to deal with them (successfully by chance or as usual, we'll just see!

So let's start giving names. For whom shall we surely find a name?

There are undoubtedly some 2 heroes:

1: the day's portion of the (every) dragon – let's denote that not so small portion by ☻, and also

2: the portion or quantity of water the quell is producing (also day by day) by ♠.

There is also the third remarkable magnitude – that magnitude being the quantity of water in the lake we are actually dealing with – let us apply for that a highly symbolic name ∞.

Now it is high time to rewrite all that was given and what is expected to reveal.

That 183 dragon-brothers would arrange all the water ∞ within one day will look like follows

$$\infty \text{ is as much as } 183 ☻ \text{ plus } ♠$$

or

$$183 ☻ = \infty + ♠,$$

and similarly it is also announced that 37 dragon-brother would arrange the whole drinking entertainment in 5 days, or that

$$37 \cdot 5 ☻ = \infty + 5♠,$$

which is the same as

$$185 ☻ = \infty + 5♠.$$

Now we are excellently aware that we are being asked for some number in parenthesis denoted by *M* or

$$M\cdot ☻ = \infty + M\cdot♠.$$

This is not so hard to achieve. Subtracting the first from the second we get

$$2 ☻ = 4 ♠,$$

or

$$☻ = 2 ♠,$$

theoretical meaning of which in human language is that the dragons portion is equal to two day lake's supplies brought by quells.

Now in order to achieve the wanted equality

$$M\cdot ☻ = \infty + M\cdot♠$$

it is enough to the equality

$$185 ☻ = \infty + 5♠ \qquad (1)$$

to add the equality

$$☻ = 2 ♠,$$

multiplied by 180 or

$$180 ☻ = 360 ♠, \qquad (2)$$

because exactly then the coefficients by ☻ and ♠ will both be equal – and that equal number will again produce something, what would appear to be also highly symbolic!

So after these promises and good intentions adding (1) + (2) we finally get

365☻=185☻+180☻=∞+5♠+360♠=∞+365♠,

which gives that the only dragon-brother will drink out the water of lake during 365 days or exactly in one year.

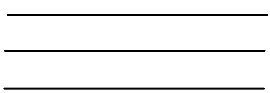
The answer.

One dragon-brother will drink out the whole lake till the bottom or the last drop of water in one normal year or in 365 days.

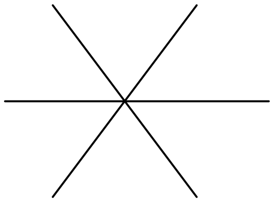
A.D. 2000, Grades 5, 6 and even 7, problem 2. King Arthur’s project to find out definitely in how many parts the plane might be split by 3 different lines of as it was investigated by the Scientific Council of His Majesty. Solution together with some commentaries added

After some experiment it is more than easy to state that 3 lines **might** divide the plane into 4 parts. This happens when

(☼) those 3 lines are all parallel as it is indicated in the drawing below

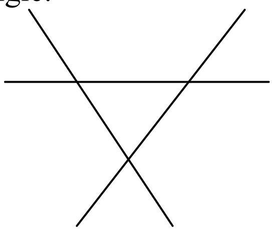


(☺) It is also possible to get even 6 parts when all these 3 lines are meeting at one points – you may again take a glimpse at the picture below.



But even 6 pieces is not the top extreme case – that extreme case

(&) are 7 pieces and it takes place when the “interior area” forms a triangle:



It may say just as Scientific Council did that it is possible to get the case (&) from the case (☺) by the some soft shift of one of these 3 lines “from their meeting point.

All what now remains to do is to prove that there are no other possible cases.

That is not so difficult to establish.

For that we firstly remove one of these 3 participating lines. Then only 2 participating lines remain and they

(♪) either divide the plane in 3 parts (being parallel) or (main case) when they are not parallel and then divide that plane into

(♫) 4 parts – when they intersect, as it was mentioned.

And now we are returning that “removed” line back.

Then if it had been 3 parts (2 parallel lines) and that comeback line is parallel to them, then it is exactly the case (☼).

If that comeback line isn't parallel to these 2 which are parallel, then (**add case**!) they all together divide that plane into 6 parts.

If the comeback happens in the situation when 2 lines made already 4 pieces and the returned line passes through that point of their intersection, then there will be again 6 parts (only in the other configuration).

And finally – if that comeback line doesn't pass through that point of intersection then 3 additional parts are being added to these existing 4 making total 7 of them.

The answer granted by Scientific Council and presented to King Arthur was:

3 lines in the plane can split it in 4, 6 or 7 parts.

A.D. 2000, Grades 5, 6 and even 7, problem 3. What a smallest number may Sir Lancelot get taking the multiples of 23 and calculating the sums of their digits?

Solution and some simplest reflections around it

Sir Lancelot was very clever and understood that having a quick glimpse at some multiples of 23 would do no harm and would be very useful – practice first of all.

So he wrote at once a dozen of those first multiples, being numbers

23, 46, 69, 92, 115, 138, 161, 184, 207, 230, 253,

with their sums of digits being

5, 10, 15, 11, 7, 12, 8, 13, 9, 5, 10.

He stated that 23 possesses the smallest sum of digits 5, and the distribution of another sums is rather chaotic. Seeking to get the smallest sum we might have some plans to get 1 as a sum. Without any doubts that sum or 1 would be the smallest possible sum of any natural number with any given property – the universal smallest number stated Sir Lancelot. But what numbers have the sum of digits 1? These are of course the numbers

1, 10, 100, 1000,….

Neither of them are divisible by 23 because these numbers are divisible only by 2's and 5's or by their products.

So it would be impossible to get 1 as a digital sum of some multiple of 23.

Further Lancelot's perspectives were connected with the hope to get 2 as a sum of digits of some multiple of 23. Being diligent by the nature and not afraid of any amount of work he wanted to start writing them down in order hoping that such a sum will appear – sooner or later. But if not, what's then? Or even if it appears but after a year, what's then? No one possesses so much time even if one has enough patience.

Lancelot understood perfectly that the sum of digits 2 have the numbers

2, 20, 200, 200,…,

as well as the numbers

11, 101, 1001, 10001, ….

(possibly with zeroes afterwards).

The history of numbers "2 with some zeroes afterwards" is very much the same a history of "1 with some zeroes afterwards" – they are also divisible by some quantity of 2's and 5's and by their products – so again there is no place for the divisibility by 23. AND THEN sir Lancelot did the main step in order to save his time and energy in order not to write without any end these multiples of 23 hoping to get some with the sum 2 in form of “many zeroes between some two 1’s”. Technically his idea was to take a number “1 with zeroes afterwards” and waiting in a long division until the intermediate rest will (if ever) be 16. Why 16? Because exactly then it is the high time to take another 1 after all these zeroes getting the number

10000000….1

which is divisible by 23, and all this is demonstrated by that long division and based on the modest fact that 161 is divisible by 23 being 7 times as much as it.

Realizing it Sir Lancelot got very soon what he wanted to get – see his notation of long division:

```
100000000001   |23
 92             4347826087
  80
  69
  110
   92
   180
   161
    190
    184
      60
      46
      140
      138
        200
        184
         161
         161
           0
```

indicating that

$$100000000001 = 23 \cdot 4347826087 .$$

We state that the wisdom of Sir Lancelot was very well based because otherwise writing down multiples of 23 he practically wouldn't have so much time as to write

4 347 826 087

numbers, which is already comparable with the number of human population of our globe – or almost one multiple for each of us living on the Earth!

The (Sir Lancelot's) answer:

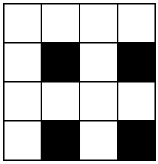
the smallest possible sum of digits of a number, which is multiple of 23, is exactly 2.

A.D. 2000, Grades 5, 6 and even 7, problem 4. That broken 2×2 tile with following attempts to replace it by 1×4 tile or around the deeds of Melvin and considerations of Pinocchio

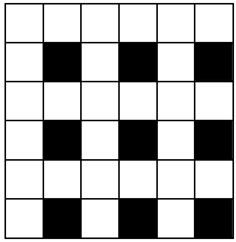
Pinocchio must have mentioned and at least clearly felt something because he, optimist as he always has been, remains sceptic about that replacing possibility. Pinocchio must have seen something what remains meanwhile hidden for us.

First of all 2000×2000 is a bit too large and in order to simplify it Pinocchio proposed to replace that 2000×2000 by the 6×6 or even by 4×4 box. He is explaining for all that there's not any essential difference between these two and the initial 2000×2000 case. Let's listen to him for a while.

And now the main idea or demonstration of certain circumstances, which made Pinocchio to remain sceptic while Melvin still remained optimistic. Pinocchio presented two pictures with “each fourth” entry shaded in a regular way. It is demonstrated in two pictures below.



(◊)



(☺).

These shaded entries form a kind of skeleton. In the case (◊) that skeleton consists of 2×2 or 4 and in the case (☺) – of 3×3 or 9 entries. In (◊) the bottom would be covered by 4 tiles. Assume that there are ♣ tiles of the size 2×2 and ♥ tiles 1×4, which we can also rotate (and to refer also as to 4×1 tiles).

	♣	♥
	4	0
	3	1
Possible number of applied tiles of each sort	2	2
	1	3
	0	4

Actually the solution now is fully understandable if you are not afraid to read the following sentence, if necessary, twice or even thrice.

Each 2×2 tile covers exactly 1 entry of our skeleton and that of 1×4 (or 4×1) covers either 0 or 2 entries of that skeleton, and exactly this makes the single replacement of any tile of one sort by tile of other sort impossible.

Let us, for example, try to imagine the case when three 2×2 tiles and 1×4 tiles completely cover the bottom of our box **AND SAY "NOTICE THAT SUCH COVERING IS IMPOSSIBLE. WHY?"**

This is impossible because each 2×2 tile covers exactly 1 shaded entry of that skeleton and that of 1×4 covers either 0 or 2 such entries. If the covering were possible then these three 2×2 tiles together with 1 tile of size 1×4 would cover either 3 or 5 tiles of the skeleton. But that skeleton consists of exactly 4 entries.

It means in our case, if the covering is complete then there must be even number of 1×4 (together with 4×1) tiles.

But then replacement is indeed impossible because it always changes the parity, and this is bad.

All other cases are similar – so, for example, in the 6×6 case we would necessarily have odd number of 2×2 and even number of 1×4 tiles – making again no single replacement impossible!

The answer.

Pinocchio is right and Melvin – not as much!

THE THIRD LITHUANIAN MATHEMATICAL OLYMPIAD FOR YOUNGSTERS (2001)

Grades 5 and 6

1. WINNIE AND 2 KILOS IN 3 WEIGHTINGS

Winnie-the-Pooh came into possession of a sack containing 9 kg of raisins as well as the balances with 2 plates and one 200 g weight. He woke up early in the morning with the bold project thundering in his head: in three weightings to get 2 kg of raisins for his coach Piggy (don't mix him with Piglet!)

2. THE SINGLE MISTAKE OF MACAVITY THE MYSTERY CAT

Using telegraph and his extraordinary skills the Macavity, the Mystery Cat, is able to transfer the following signs:

0, 1, 2, 3, 4, 5, 6, 7, 9, +, – , · (multiplication sign), : (division sign) and =(equality sign).

Being excited when transferring the correct equality Macavity was once mistaken and so the clearly incorrect equality

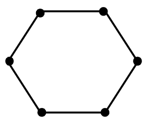
$$9 \cdot 5 + 1045 = 1990$$

was transmitted.

What could the original equality have been?

Try to indicate all possible answers.

3. ROBINSON CRUSOE AND END POINTS OF 6 LINE SEGMENTS



Once Robinson Crusoe located 6 line segments in such a way, that they have 6 endpoints all together in common.

Feeling involved he asked Man Friday to look for some other possibilities, e.g. whether it is possible that the 6 line segments would have 7 endpoints in common.

Man Friday, the fundamental worker, demonstrated that this is possible and also established all other possible cases for those 6 line segments. What are these other possibilities for 6 line segments? How many endpoints might they have in common?

4. TOM AND JERRY WITH THEIR ATTEMPTS

Deep in their hearts Tom and Jerry are always eager to improve that world changing everything in it to best till the smallest details.

So now getting the following 3×3 table with + and – signs in each of its 9 entries, which looks as follows:

+	+	–
–	+	+
+	–	+

Tom overtook the responsibility for the changes in its rows and Jerry – in its columns. Tom is allowed to choose any row as well as Jerry any column and change all three its signs in that row – as well as Jerry in any chosen column – by opposite ones. Both of them might repeat their operations as many times they wish in arbitrary order.

Their bright dream and everlasting hope is, applying that procedure and acting together, to make all entries in that table entirely positive.

+	+	+
+	+	+
+	+	+

Assuming that Tom and Jerry indeed can take for this goal enough time, do they have any chances to realize their ambitious project?

Grades 7 and even 8

1. TWO AIMS OF FAMOUS SANCHEZ PONCHO

Sanchez Poncho hopes to meet a number, which is not only written using exceptionally the digits 3 and 7 only, but moreover the sum of its digits ought also to be divisible by these digits 3 and 7 as well.

- ☻ Help Sanchez Poncho to indicate at least one such positive integer;
- ◙ Cooperating with him for a longer time if necessary, make all of us truly believe that Sanchez Poncho has correctly indicated the smallest such integer.

2. ALICE AFTER WONDERLAND AND WHAT'S THEN

Only few persons ever heard that after her adventures Alice became a professor for arts and math. That below was her favorite equation which she always proposed to those listeners who wanted to know whether they are smart in doing simple things. Under these circumstances the following equation

$$x - y = x^2 + xy + y^2$$

appeared and the proposal to present to her all pairs (x, y) of integers x and y satisfying that equation followed, and all that happened in the following way:

- ☼ as the prelude Alice asked to present her any pair (x, y) of integers x and y satisfying it;
- **&** afterwards if such solution was presented during an hour then she continued with the question "Are there still any more solutions?"

◙ in a case if the solver after some hours was still managing the situation, the final request “could you probably find all pairs (x, y) of integers x and y satisfying that equation?” followed.

So our task is to find all pairs (x, y) of integers x and y satisfying the given equation.

3. BILLY BOY AND HIS CONVEX HIMALAYA QUADRANGLE

Billy Boy wandering through the world remains always curious about the geometrical investigations.

Once upon a time being high in Himalaya Mountains he clearly noticed on some rock the carved convex quadrangle $ABCD$ with equal angles DAB and ABC.

Moreover the length of the side BC was exactly 1 and that of AD – exactly 3.

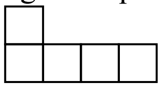
Billy Boy believes and wishes to demonstrate to all us, that the side CD is always extending 2.

Do you believe that too and could you, if needed, even consult the Billy Boy with your advices and observations?

4. BILLY BOY AND ANOTHER ROCK

On another rock, which was as smooth as only a sheet of paper might be, Billy Boy found the square 4×4 with 16 entries:

In the surroundings of that square Billy Boy also noticed the figure, which he called “*prolonged corner*” and which was made of 4 unit cells exactly of the same size as the entries of Original Square were:



Seeing that precious material Billy Boy in 2 days invented two interesting problems:

First of them – or (Ξ) – Billy Boy kindly addressed to all freshmen and another one (Ω) – to the intimate friends of combinatorics.

In (Ξ) freshmen – with all of us – were being asked the following:

At least how many cells of the original 4×4 square would be enough to shade so that in each prolonged corner (possibly rotated) at least one shaded unit cell could be found?

In (Ω) the intimate friends were being asked
the same question with the only difference that now it was allowed not only to rotate but also to turn over that “prolonged corner”.

SOLUTIONS AND AROUND THEM

A.D. 2001, Grades 5 and even 6, problem 1. Winnie’s efforts to get 2 kilos in 3 weightings

Starting the solution the first thought of Winnie was the following one – oh, how would all this be if it was possible to undertake 10 weightings! Having a weight of 200 g to get 2000 g or 2 kg in 10 weightings is just as difficult as one times one. But with only 3 weightings allowed it would be more interesting.

So he started with the whole amount of 9 kg of raisins dividing them into “two almost equal parts” using that 200 g weight on the left and involving all raisins making both sides equal. It was easy to conclude that making both sides equal and creating the situation

raisins with 200 g weight on left side being the same as only raisins on the right side

and involving all 9 kg of it, after that very first if successful weighting Winnie will get

(♪) 4400 g

from the left side together with

(♫) 4600 g

appearing from the right.

Both 4400 g and 4600 g weight only a little bit more than 2 times 2 kg so use the balance in the same way as before with the total amount of raisins being now either 4400 or 4600 g.

Winnie would get from the left side

2100 g dividing that total amount of

(♪) 4400 g

and

2200 g dividing the total amount of

(♫) 4600 g

if we would again split these mentioned quantities into “two almost equal parts” just as we did before. Naturally from the right the “complementary” weights or those of 2300 and 2400 g would appear.

But in the second case having 2200 g we are indeed one step from our aim – and that one step will be the third final weighting – we will put 200 g weight on the left and take some raisins from these 2200 g until both sides would be equal. So we would take away exactly 200g from these 2200 g and after that 3 weightings in that sack with 2200 g just

$$2200 - 200 = 2000 \text{ g}$$

or exactly 2 kg raisins for beloved coach Piggy will remain.

The answer.

The bold & bright idea of Winnie is realizable. For details see again the solution if necessary.

A.D. 2001, Grades 5 and even 6, problem 2. The single mistake of Macavity, the Mystery Cat. Solution and clear things around it

Macavity being the Mystery Cat applied the diligent idea: looking at that incorrect equality

$$9 \cdot 5 + 1045 = 1990$$

which is as close as possible to the correct one and which contains exactly 13 signs one might try to replace each of these 13 signs in all possible ways using these allowed 15 signs.

Realizing it and working hard Macavity found the following 3 possibilities:

(♠) replacing the sign of multiplication · by 4 we'll get a correct equality

$$945 + 1045 = 1990\,;$$

(♣) replacing the first 9 in the number 1990 by 0 we'll get again a correct equality

$$9 \cdot 5 + 1045 = 1090$$

and also

(♦) replacing the second digit 0 of the number 1045 by 9 we'll also get a correct equality

$$9 \cdot 5 + 1945 = 1990$$

The answer.

Macavity, the Mystery Cat, is able to get 3 correct equalities from the incorrect one by changing only one sign of it.

A.D. 2001, Grades 5 and even 6, problem 3. Man Friday selecting "as many end points as 6 line segments might share?" Around the solution

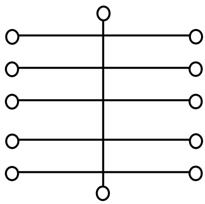
Clearly 6 line segments cannot possess more than

$$6 \cdot 2 = 12$$

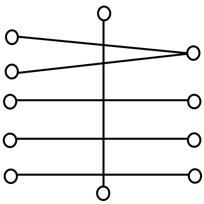
end points. So 12 is clear upper bound for 6 line segments.

And what about the lower bound? Clearly 3 points are too few because then there might be only 3 different line segments ending in these points – just as it is in every non-degenerated triangle.

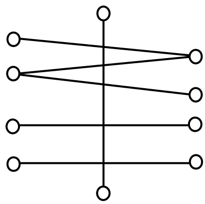
All other cases are realizable – see the drawings below where all these cases from 4 till 12 are being shown – in another order.



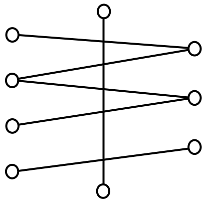
(12 end points)



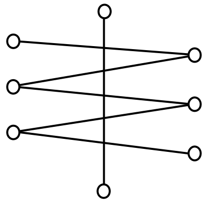
(11 end points)



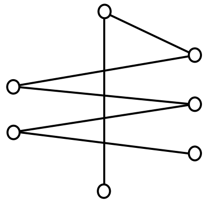
(10 end points)



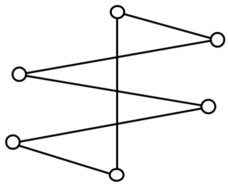
(9 end points)



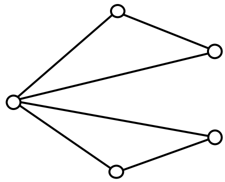
(8 end points)



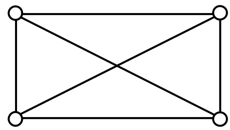
(7 end points)



(6 end points)



(5 end points)



(4 end points)

The answer.

6 line segments might “end in” 4, 5, 6, 7, 8, 9, 10, 11, 12 points.

A.D. 2001, Grades 5 and even 6, problem 4. Serious-minded Tom and Jerry with their projects to make in that table all entries positive changing signs in rows and columns. Around the solution and their perspectives

It’s such a pity to announce that the answer is ***no****!*

This project can’t be realized and entries of the table cannot be made to be all positive at a time.

This is so because in the corners of that given table

+	+	–
–	+	+
+	–	+

we have 3 entries with “+ “and 1 with “-“ sign. And now all understanding of the essence of what’s happening is contained in that sentence:

After each operation in all 4 corners we will always have either:

(♪) 3 entries with “+” and 1 with “–” signs

or else

(♫) 3 entries with “–” and 1 with “+” signs

Nothing more is possible to achieve (in corners!)

It means that all attempts to make all “+” signs in each of 9 entries will fail because in 4 corners we would enjoy 4 “+”signs then.

But as it was just stated – 4 pluses in 4 corners – that’s not our case.

In other words this is impossible because this is impossible already in corners only.

The answer.

No, even serious-minded Tom and Jerry cannot make all entries in that table positive.

A.D. 2001, Grades 7 and even 8, problem 1. Sanchez Poncho task and reality. Solution and related details

Any fan of googol-type numbers including Sanchez Poncho and all of us within the advisory board wouldn’t ever have a slightest doubt that, e.g. the following 42-digit number containing 21 sevens and 21 threes or

777 777 777 777 777 777 777 333 333 333 333 333 333 333

with sum of digits being

$$21 \cdot (3 + 7) = 210$$

would be indeed divisible by both digits 3 and 7 and is one from the possible answers for the Part ☻.

Some much smaller number of the type

7 773 333 333

might also be selected..

◙ It remains to detect the smallest such integer.

We imagine that we just hear the rough voice of Sanchez Poncho explaining as if for all of us standing around the current wind mill with Him standing in the front:

Imagine we have just met such a number. That honorable number contains

☺ 3's

as well as

☼ 7's.

I understand – Sanchez Poncho continued – that both types of digits are presented. Then the total sum of digits of that number being

$$3\cdot☺ + 7\cdot☼$$

must, as told, be divisible by 3 and also by 7 as well.

Now, $3\cdot☺$ is clearly divisible by 3; it follows then that also the number

$$(3\cdot☺ + 7\cdot☼) – 3\cdot☺ = 7\cdot☼$$

is divisible by 3, so that the number

☼

is also divisible by 3.

Similarly the number

☺

is divisible by 7.

So now we've demonstrated that the smallest such number must contain at least seven 3's and three 7's or at least

$$7 + 3 = 10$$

digits

Clearly the number

$$3\ 337\ 777\ 777$$

will resolutely be the smallest among all of them and so will serve as an answer.

The answer.

Sanchez Poncho absolutely correctly indicated that the smallest of numbers wanted is the 10-digit number

3 337 777 777.

A.D. 2001, Grades 7 and even 8, problem 2. Alice after Wonderland with her equation. Around the solution

We remind that the task was to solve equation

$$x - y = x^2 + xy + y^2$$

presenting at first any one solution; then, if possible, some 2; and finally all pairs (x, y) of integers x and y satisfying it.

First step is simple: we notice that if both x and y vanish then nothing remains on either sides – but that exactly means that the pair

$$(0; 0)$$

is a solution of given equation. That simple observation successfully completes the first or introduction part ☼ of Alice's challenge.

The second step would be when only one of those heroes vanishes while the other remains as it was:

If x vanishes and y remains as it was then

$$-y = y^2$$

This leads either to

$$y = 0$$

or

$$y = -1$$

and so we get two pairs:

(0; 0) (this is our previous pair)

and the new pair

(0; –1).

Having two solutions we've completed also the middle Part **&**, and now we are expected present to our Alice some general method how to detect all possible integer solutions.

And now, if we seek to find all solutions, then we must understand that with "pure guessing" we'll never be sure not to omit some single pair of suitable integers x and y.

Where do the boundaries of guessing lay?

Roughly speaking (being finite) we can't check infinitely many solutions simply plunging particular numbers into that equation. Because we have not infinitely many time. So we must try to solve it in general or to invent some "finite" procedure which would lead us to "all solutions".

And to our astonishment that's possible because at that very moment the best student of Alice – that best student being of course Billy Boy – appeared. And there's no need to say that after considerable efforts Billy did it. He – our hero unnamed – did it! Afterwards there were some speculations – maybe somebody from the Advisory Board consulted him. This is not so important. The main thing is that Billy Boy understood it. Simple boy. And not so very simple boy.

Technical means he demonstrated by that were far from astonishing – only multiplying both sides by 2 with the following "square making" and the final conclusion, which was entirely common sense proposal. Nothing more. Just take a look. So first multiplication by 2 or writing that

$$2x^2 + 2xy + 2y^2 - 2x + 2y = 0$$

Now, as told, that square making followed

$$x^2 + 2xy + y^2 + x^2 - 2x + 1 + y^2 + 2y + 1 = 2$$

leading to

$$(x + y)^2 + (x - 1)^2 + (y + 1)^2 = 2$$

with the clear common sense conclusion:

Assuming x and y to be integer numbers and the sum of three squares to be equal to 2 it must be then that some two of these squares are 1 and the third is 0. This means that two expressions within brackets ought to be

$$\pm 1$$

and 0 ought to appear within the third brackets.

If $\begin{cases} x + y = 0, \\ x - 1 = \pm 1, \\ y + 1 = \pm 1, \end{cases}$ then we get pairs (0; 0) and (2; –2)

If $\begin{cases} x + y = \pm 1, \\ x - 1 = 0, \\ y + 1 = \pm 1, \end{cases}$ then we get pairs (1; 0) and (1; –2).

If $\begin{cases} x + y = \pm 1, \\ x - 1 = \pm 1, \\ y + 1 = 0, \end{cases}$ then we get pairs (2; –1) and (0; –1).

The answer.

These are all in all 6 solutions or

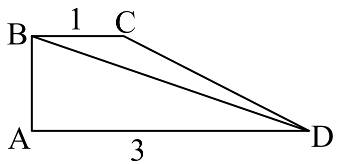
(0; –1), (0; 0), (1; –2), (1; 0), (2; –2), (2; –1)

of that famous equation proposed by Alice as they were found in our eyes by Billy Boy – we all were witnesses of that!

A.D. 2001, Grades 7 and even 8, problem 3. Billy Boy and his convex Himalaya quadrangle. Around the solution

Let us cite probably the most precious refrain when singing or otherwise performing on the triangles stage:

An angle lying opposite to the bigger side of triangle is also the bigger angle, and inversely.



After repeating that refrain if necessary even thrice – it's surely worth repeating – we've started to check Billy Boy's computations. We're very much impressed understanding how effectively he was performing – of course it wasn't very difficult.

But nevertheless it was a good job. So he continued

The angle A (former angle DAB) in the triangle ABD is bigger than the angle B (being the part of angle ABC, which is as big as DAB). This explains why the side BD opposite to A is longer than the side AD which is 3. So

$$BD > 3.$$

Another refrain on the triangle stage known for all who attended to at least three geometry lessons sounds:

Every single side of a triangle is shorter than the sum of the lengths of the remaining two sides.

In our case having in mind the triangle BCD we might write

$$BC + CD > AD$$

or

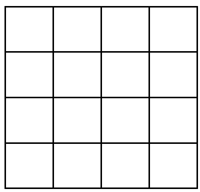
$$CD + 1 > AD > 3$$

ensuring that

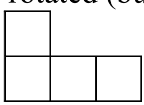
$$CD > 2\,.$$

as needed.

A.D. 2001, Grades 7 and even 8, problem 4.
Billy Boy, another rock and the freshmen or about how many cells is it enough to shade in order that in any (probably rotated) "prolonged corner" it would be at least one shaded 1×1 cell.



Let us repeat that "*prolonged corner*" is made of 4 cells, which are exactly of the same size as the entries of Original Square. It looks like it is shown below. Naturally it can also be rotated (but still not overturned!)



Let us notice that that “*prolonged corner*” is very similar to the Greek letter Γ.

Firstly, we will state the fundamental thing the importance of which is immense, or say the following bold but right fact:

Bold but right statement:

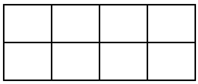
Billy Boy, in order to solve that problem for freshmen, is forced to shade at least 4 cells.

How to prove it?

It is very simple – because you always can divide the 4×4 square into 4 “prolonged corners”.

There is no need even to describe in which way it could be done – this is clear. Or it could be added that each

2×4



rectangle we can split into two (rotated) “*prolonged corners*” or Greek letters Γ (also, of course, rotated).

So if our 4 × 4 square consists of exactly 4 nonoverlapping “prolonged corners” then in each such “prolonged corner” at least one cell must be shaded.

It means that it is absolutely necessary to shade 4 cells.

Now it remains to demonstrate that 4 shaded cells are enough.

We are forced to find some universal example – but how could we guess that this is possible?

We couldn’t do it so quickly but Billy Boy one day found it.

You wouldn’t believe it if you couldn’t see it.

But it is possible and we also in Advisory Board enjoyed that example of Billy Boy. Till this day we do not exactly understand in what way he came to it. But we have seen it with our own eyes and moreover we are ready to demonstrate it also for you. You may also enjoy it.

Below we demonstrate the universal example of Billy Boy demonstration that to shadow 4 cells is enough. As usual the shaded cells will be marked by X.

		X	X
X	X		

And no "*prolonged corner*" (possibly rotated) in the form of Greek letter Γ may avoid some of these 4 shaded cells.

We remind that "*prolonged corner*" may be rotated but not turned over.

The problem with "*prolonged corner*", which could not only be rotated but also turned over, is already the advanced problem of Billy Boy for advanced solvers.

Now it will be probably a bit more complicated to find the answer. It is clear that the previous famous example of Billy Boy is not valid because if we are allowed to turn over the prolonged square then in that example we can find plenty of "*prolonged corners*" so similar to that Greek letter Γ.

In that place when it is already permitted not only to rotate "*prolonged corners*" but also to turn them over, Billy Boy demonstrated another example showing that 5 shaded cells is enough – you again may take a look to the picture!

	X		
		X	
X			X
	X		

It remains to see that 4 shaded cells are not enough.

Let us examine the non-shaded original square

We will prove that 4 shaded cells are not enough.

Assume it is enough. We intend to prove that all border cells remain non-shaded.

Firstly we'll prove that all the cells of the lowest row remain non-shaded.

The proof that all cells of highest row as well as all cells of upright and also upleft column must also be non-shaded if 4 cells is enough remains very similar.

So we would like to prove that all cells of the lowest row remain non-shaded.

Consider the 3×2 rectangle of that square containing the top-left corner cell (in chess notation a2b2b4a4 rectangle).

In that rectangle there must be at least 2 shaded cells because otherwise we will build a prolonged corner in it.

Similarly regarding another 3×2 rectangle containing the top –right corner cell (in chess notation c2d2d4c4 rectangle) we conclude:

In that rectangle there must be also at least 2 shaded cells because otherwise we will build a prolonged corner in that rectangle.

But then in 3 highest rows we will find already all 4 shaded cells.

So the lowest row is non-shaded.

Similarly, also the highest row – modifying what was just told!

And also the leftmost column as well as rightmost one are also non-shaded.

So all border cells are non-shaded.

But then many prolonged corners at the border with non-shaded cells may be demonstrated.

The answer.

For freshmen 4 cells to shadow are enough.

Example for freshmen (no turnovers are allowed, only rotations):

		X	X
X	X		

Example for intimate friends of Billy Boy shows that 5 shaded cells are enough (turnovers are allowed):

	X		
		X	
X			X
	X		