The LAIMA series

Matti LEHTINEN

THE NORDIC MATHEMATICAL COMPETITION 1987 - 2006

Problems and Solutions

Rīga 2006

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The Nordic Mathematical Competition (NMC) is a contest for secondary school students in solving mathematics problems on a quite high level of difficulty. NMC is organized in five Nordic countries: Denmark, Finland, Iceland, Norway, and Sweden.
This collection contains the problems and solutions of the first 20 NMC's.

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About the LAIMA series

In 1990, the international team competition "Baltic Way" was organized for the first time. The competition gained its name from the mass action in August, 1989, when over a million people stood hand in hand along the Tallinn – Riga – Vilnius road, demonstrating their will for freedom.

Today "Baltic Way" has all the countries around the Baltic Sea (and also Iceland) as its participants. Inviting Iceland is a special case remembering that it was the first country in the world which officially recognized the independence of Lithuania, Latvia and Estonia in 1991.

The "Baltic Way" competition has given rise to other mathematical activities, too. One of them is the project LAIMA (Latvian – Icelandic Mathematics Project). Its aim is to publish a series of books covering all essential topics in the arena of mathematical competitions.

Mathematical olympiads today have become an important and essential part of the education system. In some sense they provide high standards for teaching mathematics on an advanced level. Many outstanding scientists are involved in composing problems for competitions. The "olympiad curriculum", considered all over the world, is a good reflection of important mathematical ideas on elementary level.

It is the opinion of the publishers of the LAIMA series that there are relatively few important topics which cover almost everything that the the international mthematical community has recognized as worthy to be included regularly in the search and promotion of young talent. This (clearly subjective) opinion is reflected in the list of teaching aids which are to be prepared within the LAIMA project.

Seventeen books have been published so far in Latvian. They are also electronically available in the web page of the Latvian Education Information System (LIIS), `http://www.liis.lv`. As LAIMA is rather a process than a project, there is no idea of a final date; many of the already published teaching aids are second or third versions and they will be extended regularly.

Benedict Johannesson, President of the Icelandic Society of Mathematics, gave inspiration to the LAIMA project in 1996. Being a co-author of many LAIMA publications, he also was the main sponsor for many years.

This book is the third LAIMA publication in English. It was sponsored by the Scandinavian "Nord Plus Neighbours" foundation.

FOREWORD

The Nordic Mathematical Competition, NMC, has its roots in the International Mathematical Olympiads, IMO's. In the 1986 IMO in Warsaw the leaders of the Nordic teams realized that one reason behind the rather mediocre if not bad success of their teams was lack of competition experience at a more difficult level. The countries did not possess a large-scale multistage national mathematical competition, and the existing competitions were rather easy. As a remedy to the situation, a cheap and easily manageable competition was proposed, and the NMC has been arranged every year since 1987. This means that there are now 20 problem sets of the NMC, and to make them available seems to be appropriate.

The way the NMC is run has remained unchanged. The five participating Nordic countries, Denmark, Finland, Iceland, Norway, and Sweden, alternate as the host or organizing country. Each country has a contact person responsible for the management of the competition in her or his country. The organizer solicits problem proposals from the other countries, prepares the problem sheet consisting of four problems. The level of the problems is moderate, clearly below the IMO difficulty. The text is accepted by the other countries and translated into the five languages used in the countries. Each country is allowed to enroll 20 participants. They are students considered to be possible candidates for the IMO team, and in each country, the NMC is one of the main criteria used in selection of the team. Eligibility criteria thus are the same as in the IMO: the participants are secondary school students and less than 20 years old. The competition takes place in March or April. Each student does the problems in her or his own school under the school's supervision. The time allowed is four hours. The schools have never refused their cooperation. The answers are marked preliminarily in each country, and then sent, together with necessary translations, to the organizing country, which coordinates the marking. The results and diplomas – always in the language of the organizing country – are ready to be mailed before the end of the school semester in May.

This collection contains the problems and solutions of the first 20 NMC's. As quite a number of people have been involved in creating and choosing the competition problems in this period, the compiler has been able to utilize the fruits of much collective work. In most cases, the solution ideas go back to the origal proposers.

The problem texts of the NMC's have always been prepared in English, but in the preparation of this booklet, not all of these texts were available. The majority of the problems have been translated from the Finnish problem sheets. This may cause minor differences to the "official texts", which, on the other hand, have not actually been used in the competitions, as the competitors have worked in Danish, Finnish, Icelandic, Norwegian, or Swedish. Sometimes explanatory notes have been included in the problem texts. These have been preserved, although they sometimes seem to be unnecessary. Also, there is some variation in notation and in the ways some words have been italicized in the problem texts. The original notation and italicization have been preserved.

The solutions sometimes utilize standard abbreviations like "sas" for the theorem (or axiom) on the congruence of triangle with two pairs of equal sides and an equal angle between them. The notations used are standard. Results referenced to in the solutions are those one usually meets in the field of "olympiad mathematics". Some of these are rather distant from the usual school curriculum.

A good competition problem often can be approached from a number of different angles. The solutions in this booklet are by no means the only possible ones. In some instances, alternative solutions are given, but anyone trying these problems should delight himself when finding another, correct solution. The compiler of this collection is happy to receive any such solutions, for instance to to his email address, `matti.lehtinen@helsinki.fi`.

Helsinki, Finland, August 2006
Matti Lehtinen

PROBLEMS

NMC 1, March 30, 1987

87.1. Nine journalists from different countries attend a press conference. None of these speaks more than three languages, and each pair of the journalists share a common language. Show that there are at least five journalists sharing a common language.

87.2. Let $ABCD$ be a parallelogram in the plane. We draw two circles of radius R, one through the points A and B, the other through B and C. Let E be the other point of intersection of the circles. We assume that E is not a vertex of the parallelogram. Show that the circle passing through A, D, and E also has radius R.

87.3. Let f be a strictly increasing function defined in the set of natural numbers satisfying the conditions $f(2) = a > 2$ and $f(mn) = f(m)f(n)$ for all natural numbers m and n. Determine the smallest possible value of a.

87.4. Let a, b, and c be positive real numbers. Prove:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \le \frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}.$$

NMC 2, April 4, 1988

88.1. The positive integer n has the following property: if the three last digits of n are removed, the number $\sqrt[3]{n}$ remains. Find n.

88.2. Let a, b, and c be non-zero real numbers and let $a \geq b \geq c$. Prove the inequality

$$\frac{a^3 - c^3}{3} \geq abc\left(\frac{a-b}{c} + \frac{b-c}{a}\right).$$

When does equality hold?

88.3. Two concentric spheres have radii r and R, $r < R$. We try to select points A, B and C on the surface of the larger sphere such that all sides of the triangle ABC would be tangent to the surface of the smaller sphere. Show that the points can be selected if and only if $R \leq 2r$.

88.4. Let m_n be the smallest value of the function

$$f_n(x) = \sum_{k=0}^{2n} x^k.$$

Show that $m_n \to \frac{1}{2}$, as $n \to \infty$.

NMC 3, April 10, 1989

89.1. Find a polynomial P of lowest possible degree such that

(a) P has integer coefficients,
(b) all roots of P are integers,
(c) $P(0) = -1$,
(d) $P(3) = 128$.

89.2. Three sides of a tetrahedron are right-angled triangles having the right angle at their common vertex. The areas of these sides are A, B, and C. Find the total surface area of the tetrahedron.

89.3. Let S be the set of all points t in the closed interval $[-1, 1]$ such that for the sequence x_0, x_1, x_2, $\ldots$ defined by the equations $x_0 = t$, $x_{n+1} = 2x_n^2 - 1$, there exists a positive integer N such that $x_n = 1$ for all $n \geq N$. Show that the set S has infinitely many elements.

89.4. For which positive integers n is the following statement true: if a_1, a_2, $\ldots$, a_n are positive integers, $a_k \leq n$ for all k and $\sum_{k=1}^{n} a_k = 2n$, then it is always possible to choose a_{i_1}, a_{i_2}, $\ldots$, a_{i_j} in such a way that the indices i_1, i_2, $\ldots$, i_j are different numbers, and $\sum_{k=1}^{j} a_{i_k} = n$?

NMC 4, April 5, 1990

90.1. Let m, n, and p be odd positive integers. Prove that the number

$$\sum_{k=1}^{(n-1)^p} k^m$$

is divisible by n.

90.2. Let a_1, a_2, $\ldots$, a_n be real numbers. Prove

$$\sqrt[3]{a_1^3 + a_2^3 + \ldots + a_n^3} \leq \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}. \qquad (1)$$

When does equality hold in (1)?

90.3. Let ABC be a triangle and let P be an interior point of ABC. We assume that a line l, which passes through P, but not through A, intersects AB and AC (or their extensions over B or C) at Q and R, respectively. Find l such that the perimeter of the triangle AQR is as small as possible.

90.4. It is possible to perform three operations f, g, and h for positive integers: $f(n) = 10n$, $g(n) = 10n + 4$, and $h(2n) = n$; in other words, one may write 0 or 4 in the end of the number and one may divide an even number by 2. Prove: every positive integer can be constructed starting from 4 and performing a finite number of the operations f, g, and h in some order.

NMC 5, April 10, 1991

91.1. Determine the last two digits of the number

$$2^5 + 2^{5^2} + 2^{5^3} + \cdots + 2^{5^{1991}},$$

written in decimal notation.

91.2. In the trapezium $ABCD$ the sides AB and CD are parallel, and E is a fixed point on the side AB. Determine the point F on the side CD so that the area of the intersection of the triangles ABF and CDE is as large as possible.

91.3. Show that

$$\frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} < \frac{2}{3}$$

for all $n \geq 2$.

91.4. Let $f(x)$ be a polynomial with integer coefficients. We assume that there exists a positive integer k and k consecutive integers n, $n+1$, $\ldots$, $n+k-1$ so that none of the numbers $f(n)$, $f(n+1)$, $\ldots$, $f(n+k-1)$ is divisible by k. Show that the zeroes of $f(x)$ are not integers.

NMC 6, April 8, 1992

92.1. Determine all real numbers $x > 1$, $y > 1$, and $z > 1$, satisfying the equation

$$x + y + z + \frac{3}{x-1} + \frac{3}{y-1} + \frac{3}{z-1} = 2\left(\sqrt{x+2} + \sqrt{y+2} + \sqrt{z+2}\right).$$

92.2. Let $n > 1$ be an integer and let $a_1, a_2, \ldots, a_n$ be n different integers. Show that the polynomial

$$f(x) = (x - a_1)(x - a_2) \cdot \ldots \cdot (x - a_n) - 1$$

is not divisible by any polynomial with integer coefficients and of degree greater than zero but less than n and such that the highest power of x has coefficient 1.

92.3. Prove that among all triangles with inradius 1, the equilateral one has the smallest *perimeter*.

92.4. Peter has many squares of equal side. Some of the squares are black, some are white. Peter wants to assemble a big square, with side equal to n sides of the small squares, so that the big square has *no* rectangle formed by the small squares such that all the squares in the vertices of the rectangle are of equal colour. How big a square is Peter able to assemble?

NMC 7, March 17, 1993

93.1. Let F be an increasing real function defined for all x, $0 \le x \le 1$, satisfying the conditions

(i) $\quad F\left(\frac{x}{3}\right) = \frac{F(x)}{2}$,

(ii) $\quad F(1 - x) = 1 - F(x)$.

Determine $F\left(\frac{173}{1993}\right)$ and $F\left(\frac{1}{13}\right)$.

93.2. A hexagon is inscribed in a circle of radius r. Two of the sides of the hexagon have length 1, two have length 2 and two have length 3. Show that r satisfies the equation

$$2r^3 - 7r - 3 = 0.$$

93.3. Find all solutions of the system of equations

$$\begin{cases} s(x) + s(y) = x \\ x + y + s(z) = z \\ s(x) + s(y) + s(z) = y - 4, \end{cases}$$

where x, y, and z are positive integers, and $s(x)$, $s(y)$, and $s(z)$ are the *numbers of digits* in the decimal representations of x, y, and z, respectively.

93.4. Denote by $T(n)$ the *sum of the digits of the decimal representation* of a positive integer n.

a) Find an integer N, for which $T(k \cdot N)$ is even for all k, $1 \le k \le 1992$, but $T(1993 \cdot N)$ is odd.

b) Show that no positive integer N exists such that $T(k \cdot N)$ is even for all positive integers k.

NMC 8, March 17, 1994

94.1. Let O be an interior point in the equilateral triangle ABC, of side length a. The lines AO, BO, and CO intersect the sides of the triangle in the points A_1, B_1, and C_1. Show that

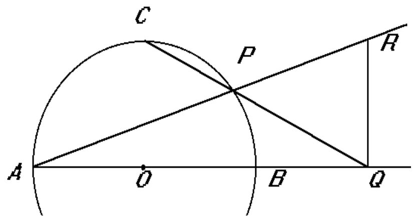
$$|OA_1| + |OB_1| + |OC_1| < a.$$

94.2. We call a finite plane set S consisting of points with integer coefficients a *two-neighbour set*, if for each point $(p,\, q)$ of S exactly two of the points $(p+1,\, q)$, $(p,\, q+1)$, $(p-1,\, q)$, $(p,\, q-1)$ belong to S. For which integers n there exists a two-neighbour set which contains exactly n points?

94.3. A piece of paper is the square $ABCD$. We fold it by placing the vertex D on the point D' of the side BC. We assume that AD moves on the segment $A'D'$ and that $A'D'$ intersects AB at E. Prove that the perimeter of the triangle EBD' is one half of the perimeter of the square.

94.4. Determine all positive integers $n < 200$, such that $n^2 + (n+1)^2$ is the square of an integer.

NMC 9, March 15, 1995



95.1. Let AB be a diameter of a circle with centre O. We choose a point C on the circumference of the circle such that OC and AB are perpendicular to each other. Let P be an arbitrary point on the (smaller) arc BC and let the lines CP and AB meet at Q. We choose R on AP so that RQ and AB are perpendicular to each other. Show that $|BQ| = |QR|$.

95.2. Messages are coded using sequences consisting of zeroes and ones only. Only sequences with at most two consecutive ones or zeroes are allowed. (For instance the sequence 011001 is allowed, but 011101 is not.) Determine the number of sequences consisting of exactly 12 numbers.

95.3. Let $n \geq 2$ and let $x_1, x_2, \ldots x_n$ be real numbers satisfying $x_1 + x_2 + \ldots + x_n \geq 0$ and $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$. Let $M = \max\{x_1, x_2, \ldots, x_n\}$. Show that

$$M \geq \frac{1}{\sqrt{n(n-1)}}. \qquad (1)$$

When does equality hold in (1)?

95.4. Show that there exist infinitely many mutually non-congruent triangles T, satisfying

(i) The side lengths of T are consecutive integers.

(ii) The area of T is an integer.

NMC 10, April 11, 1996

96.1. Show that there exists an integer divisible by 1996 such that the sum of the its decimal digits is 1996.

96.2. Determine all real numbers x, such that

$$x^n + x^{-n}$$

is an integer for all integers n.

96.3. The circle whose diameter is the altitude dropped from the vertex A of the triangle ABC intersects the sides AB and AC at D and E, respectively ($A \neq D$, $A \neq E$). Show that the circumcentre of ABC lies on the altitude dropped from the vertex A of the triangle ADE, or on its extension.

96.4. The real-valued function f is defined for positive integers, and the positive integer a satisfies

$$f(a) = f(1995), \qquad f(a+1) = f(1996), \qquad f(a+2) = f(1997)$$

$$f(n+a) = \frac{f(n)-1}{f(n)+1} \quad \text{for all positive integers } n.$$

(i) Show that $f(n+4a) = f(n)$ for all positive integers n.

(ii) Determine the smallest possible a.

NMC 11, April 9, 1997

97.1. Let A be a set of seven positive numbers. Determine the maximal number of triples (x, y, z) of elements of A satisfying $x < y$ and $x + y = z$.

97.2. Let $ABCD$ be a convex quadrilateral. We assume that there exists a point P inside the quadrilateral such that the areas of the triangles ABP, BCP, CDP, and DAP are equal. Show that at least one of the diagonals of the quadrilateral bisects the other diagonal.

97.3. Let A, B, C, and D be four different points in the plane. Three of the line segments AB, AC, AD, BC, BD, and CD have length a. The other three have length b, where $b > a$. Determine all possible values of the quotient $\dfrac{b}{a}$.

97.4. Let f be a function defined in the set $\{0, 1, 2, \dots\}$ of non-negative integers, satisfying $f(2x) = 2f(x)$, $f(4x+1) = 4f(x) + 3$, and $f(4x-1) = 2f(2x-1) - 1$. Show that f is an injection, i.e. if $f(x) = f(y)$, then $x = y$.

NMC 12, April 2, 1998

98.1. Determine all functions f defined in the set of rational numbers and taking their values in the same set such that the equation $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ holds for all rational numbers x and y.

98.2. Let C_1 and C_2 be two circles intersecting at A and B. Let S and T be the centres of C_1 and C_2, respectively. Let P be a point on the segment AB such that $|AP| \neq |BP|$ and $P \neq A$, $P \neq B$. We draw a line perpendicular to SP through P and denote by C and D the points at which this line intersects C_1. We likewise draw a line perpendicular to TP through P and denote by E and F the points at which this line intersects C_2. Show that C, D, E, and F are the vertices of a rectangle.

98.3. (a) For which positive numbers n does there exist a sequence x_1, x_2, $\dots$, x_n, which contains each of the numbers 1, 2, $\dots$, n exactly once and for which $x_1 + x_2 + \cdots + x_k$ is divisible by k for each $k = 1, 2, \dots, n$?

(b) Does there exist an infinite sequence x_1, x_2, x_3, $\dots$, which contains every positive integer exactly once and such that $x_1 + x_2 + \cdots + x_k$ is divisible by k for every positive integer k?

98.4. Let n be a positive integer. Count the number of numbers $k \in \{0, 1, 2, \ldots, n\}$ such that $\binom{n}{k}$ is odd. Show that this number is a power of two, i.e. of the form 2^p for some nonnegative integer p.

NMC 13, April 15, 1999

99.1. The function f is defined for non-negative integers and satisfies the condition

$$f(n) = \begin{cases} f(f(n+11)), & \text{if } n \leq 1999 \\ n-5, & \text{if } n > 1999. \end{cases}$$

Find all solutions of the equation $f(n) = 1999$.

99.2. Consider 7-gons inscribed in a circle such that all sides of the 7-gon are of different length. Determine the maximal number of 120° angles in this kind of a 7-gon.

99.3. The infinite integer plane $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$ consists of all number pairs (x, y), where x and y are integers. Let a and b be non-negative integers. We call any move from a point (x, y) to any of the points $(x \pm a, y \pm b)$ or $(x \pm b, y \pm a)$ a (a, b)-knight move. Determine all numbers a and b, for which it is possible to reach all points of the integer plane from an arbitrary starting point using only (a, b)-knight moves.

99.4. Let $a_1, a_2, \ldots, a_n$ be positive real numbers and $n \geq 1$. Show that

$$n\left(\frac{1}{a_1} + \cdots + \frac{1}{a_n}\right) \geq \left(\frac{1}{1+a_1} + \cdots + \frac{1}{1+a_n}\right)\left(n + \frac{1}{a_1} + \cdots + \frac{1}{a_n}\right).$$

When does equality hold?

NMC 14, March 30, 2000

00.1. In how many ways can the number 2000 be written as a sum of three positive, not necessarily different integers? (Sums like $1 + 2 + 3$ and $3 + 1 + 2$ etc. are the same.)

00.2. The persons P_1, P_1, $\ldots$, P_{n-1}, P_n sit around a table, in this order, and each one of them has a number of coins. In the start, P_1 has one coin more than P_2, P_2 has one coin more than P_3, etc., up to P_{n-1} who has one coin more than P_n. Now P_1 gives one coin to P_2, who in turn gives two coins to P_3 etc., up to P_n who gives n coins to P_1. Now the process continues in the same way: P_1 gives $n+1$ coins to P_2, P_2 gives $n+2$ coins to P_3; in this way the transactions go on until someone has not enough coins, i.e. a person no more can give away one coin more than he just received. At the moment when the process comes to an end in this manner, it turns out that there are two neighbours at the table such that one of them has exactly five times as many coins as the other. Determine the number of persons and the number of coins circulating around the table.

00.3. In the triangle ABC, the bisector of angle B meets AC at D and the bisector of angle C meets AB at E. The bisectors meet each other at O. Furthermore, $OD = OE$. Prove that either ABC is isosceles or $\angle BAC = 60°$.

00.4. The real-valued function f is defined for $0 \le x \le 1$, $f(0) = 0$, $f(1) = 1$, and

$$\frac{1}{2} \le \frac{f(z) - f(y)}{f(y) - f(x)} \le 2$$

for all $0 \le x < y < z \le 1$ with $z - y = y - x$. Prove that

$$\frac{1}{7} \le f\left(\frac{1}{3}\right) \le \frac{4}{7}.$$

NMC 15, March 29, 2001

01.1. Let A be a finite collection of squares in the coordinate plane such that the vertices of all squares that belong to A are (m, n), $(m+1, n)$, $(m, n+1)$, and $(m+1, n+1)$ for some integers m and n. Show that there exists a subcollection B of A such that B contains at least 25 % of the squares in A, but no two of the squares in B have a common vertex.

01.2. Let f be a bounded real function defined for all real numbers and satisfying for all real numbers x the condition

$$f\left(x+\frac{1}{3}\right)+f\left(x+\frac{1}{2}\right)=f(x)+f\left(x+\frac{5}{6}\right).$$

Show that f is periodic. (A function f is bounded, if there exists a number L such that $|f(x)| < L$ for all real numbers x. A function f is periodic, if there exists a positive number k such that $f(x+k) = f(x)$ for all real numbers x.)

01.3. Determine the number of real roots of the equation

$$x^8 - x^7 + 2x^6 - 2x^5 + 3x^4 - 3x^3 + 4x^2 - 4x + \frac{5}{2} = 0.$$

01.4. Let $ABCDEF$ be a convex hexagon, in which each of the diagonals AD, BE, and CF divides the hexagon into two quadrilaterals of equal area. Show that AD, BE, and CF are concurrent.

NMC 16, April 4, 2002

02.1. The trapezium $ABCD$, where AB and CD are parallel and $AD < CD$, is inscribed in the circle c. Let DP be a chord of the circle, parallel to AC. Assume that the tangent to c at D meets the line AB at E and that PB and DC meet at Q. Show that $EQ = AC$.

02.2. In two bowls there are in total N balls, numbered from 1 to N. One ball is moved from one of the bowls into the other. The average of the numbers in the bowls is increased in both of the bowls by the same amount, x. Determine the largest possible value of x.

02.3. Let $a_1, a_2, \ldots, a_n$, and $b_1, b_2, \ldots, b_n$ be real numbers, and let $a_1, a_2, \ldots, a_n$ be all different. Show that if all the products

$$(a_i + b_1)(a_i + b_2) \cdots (a_i + b_n),$$

$i = 1, 2, \ldots, n$, are equal, then the products

$$(a_1 + b_j)(a_2 + b_j) \cdots (a_n + b_j),$$

$j = 1, 2, \ldots, n$, are equal, too.

02.4. Eva, Per and Anna play with their pocket calculators. They choose different integers and check, whether or not they are divisible by 11. They only look at nine-digit numbers consisting of all the digits $1, 2, \ldots, 9$. Anna claims that the probability of such a number to be a multiple of 11 is exactly 1/11. Eva has a different opinion: she thinks the probability is less than 1/11. Per thinks the probability is more than 1/11. Who is correct?

NMC 17, April 3, 2003

03.1. Stones are placed on the squares of a chessboard having 10 rows and 14 columns. There is an odd number of stones on each row and each column. The squares are coloured black and white in the usual fashion. Show that the number of stones on black squares is even. Note that there can be more than one stone on a square.

03.2. Find all triples of integers (x, y, z) satisfying

$$x^3 + y^3 + z^3 - 3xyz = 2003.$$

03.3. The point D inside the equilateral triangle $\triangle ABC$ satisfies $\angle ADC = 150°$. Prove that a triangle with side lengths $|AD|$, $|BD|$, $|CD|$ is necessarily a right-angled triangle.

03.4. Let $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ be the set of non-zero real numbers. Find all functions $f : \mathbb{R}^* \to \mathbb{R}^*$ satisfying

$$f(x) + f(y) = f(xy\,f(x+y)),$$

for x, $y \in \mathbb{R}^*$ and $x + y \neq 0$.

NMC 18, April 1, 2004

04.1. 27 balls, labelled by numbers from 1 to 27, are in a red, blue or yellow bowl. Find the possible numbers of balls in the red bowl, if the averages of the labels in the red, blue, and yellow bowl are 15, 3 ja 18, respectively.

04.2. Let $f_1 = 0$, $f_2 = 1$, and $f_{n+2} = f_{n+1} + f_n$, for $n = 1$, $2, \ldots$, be the Fibonacci sequence. Show that there exists a strictly increasing infinite arithmetic sequence none of whose numbers belongs to the Fibonacci sequence. [A sequence is *arithmetic*, if the difference of any of its consecutive terms is a constant.]

04.3. Let x_{11}, x_{21}, $\ldots$, x_{n1}, $n > 2$, be a sequence of integers. We assume that all of the numbers x_{i1} are not equal. Assuming that the numbers x_{1k}, x_{2k}, $\ldots$, x_{nk} have been defined, we set

$$x_{i,k+1} = \frac{1}{2}(x_{ik} + x_{i+1,k}),\ i = 1, 2, \ldots, n-1,$$
$$x_{n,k+1} = \frac{1}{2}(x_{nk} + x_{1k}).$$

Show that for n odd, x_{jk} is not an integer for some j, k. Does the same conclusion hold for n even?

04.4. Let a, b, and c be the side lengths of a triangle and let R be its circumradius. Show that

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \geq \frac{1}{R^2}.$$

NMC 19. April 5, 2005

05.1. Find all positive integers k such that the product of the digits of k, in the decimal system, equals

$$\frac{25}{8}k - 211.$$

05.2. Let a, b, and c be positive real numbers. Prove that

$$\frac{2a^2}{b+c} + \frac{2b^2}{c+a} + \frac{2c^2}{a+b} \geq a + b + c.$$

05.3. There are 2005 young people sitting around a (large!) round table. Of these at most 668 are boys. We say that a girl G is in a strong position, if, counting from G to either direction at any length, the number of girls is always strictly larger than the number of boys. (G herself is included in the count.) Prove that in any arrangement, there always is a girl in a strong position.

05.4. The circle $\mathcal{C}_1$ is inside the circle $\mathcal{C}_2$, and the circles touch each other at A. A line through A intersects $\mathcal{C}_1$ also at B and $\mathcal{C}_2$ also at C. The tangent to $\mathcal{C}_1$ at B intersects $\mathcal{C}_2$ at D and E. The tangents of $\mathcal{C}_1$ passing through C touch $\mathcal{C}_1$ at F and G. Prove that D, E, F, and G are concyclic.

NMC 20. March 30, 2006

06.1. Let B and C be points on two fixed rays emanating from a point A such that $AB + AC$ is constant. Prove that there exists a point $D \neq A$ such that the circumcircles of the triangels ABC pass through D for every choice of B and C.

06.2. The real numbers x, y and z are not all equal and they satisfy

$$x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x} = k.$$

Determine all possible values of k.

06.3. A sequence of positive integers $\{a_n\}$ is given by

$$a_0 = m \qquad \text{and} \qquad a_{n+1} = a_n^5 + 487$$

for all $n \geq 0$. Determine all values of m for which the sequence contains as many square numbers as possible.

06.4. The squares of a 100×100 chessboard are painted with 100 different colours. Each square has only one colour and every colour is used exactly 100 times. Show that there exists a row or a column on the chessboard in which at least 10 colours are used.

SOLUTIONS

87.1. *Nine journalists from different countries attend a press conference. None of these speaks more than three languages, and each pair of the journalists share a common language. Show that there are at least five journalists sharing a common language.*

Solution. Assume the journalists are J_1, J_2, …, J_9. Assume that no five of them have a common language. Assume the languages J_1 speaks are L_1, L_2, and L_3. Group J_2, J_3, …, J_9 according to the language they speak with J_1. No group can have more than three members. So either there are three groups of three members each, or two groups with three members and one with two. Consider the first alternative. We may assume that J_1 speaks L_1 with J_2, J_3, and J_4, L_2 with J_5, J_6, and J_7, and L_3 with J_8, J_9, and J_2 . Now J_2 speaks L_1 with J_1, J_3, and J_4, L_3 with J_1, J_8, and J_9. J_2 must speak a fourth language, L_4, with J_5, J_6, and J_7. But now J_5 speaks both L_2 and L_4 with J_2, J_6, and J_7. So J_5 has to use his third language with J_1, J_4, J_8, and J_9. This contradicts the assumption we made. So we now may assume that J_1 speaks L_3 only with J_8 and J_9. As J_1 is not special, we conclude that for each journalist J_k, the remaining eight are divided into three mutually exclusive language groups, one of which has only two members. Now J_2 uses L_1 with three others, and there has to be another language he also speaks with three others. If this were L_2 or L_3, a group of five would arise (including J_1). So J_2 speaks L_4 with three among $J_5, \ldots, J_9$. Either two of these three are among J_5, J_6, and J_7, or among J_8, J_9. Both alternatives lead to a contradiction to the already proved fact that no pair of journalists speaks two languages together. The proof is complete.

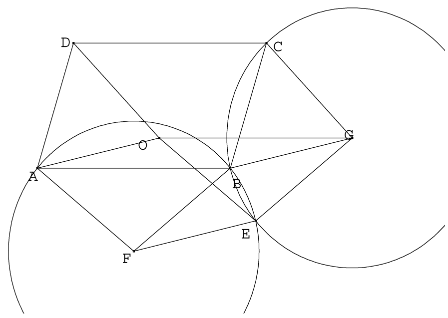


Figure 1.

87.2. *Let $ABCD$ be a parallelogram in the plane. We draw two circles of radius R, one through the points A and B, the other through B and C. Let E be the other point of intersection of the circles. We assume that E is not a vertex of the parallelogram. Show that the circle passing through A, D, and E also has radius R.*

Solution. (See Figure 1.) Let F and G be the centers of the two circles of radius R passing through A and B; and B and C, respectively. Let O be the point for which the the rectangle $ABGO$ is a parallelogram. Then $\angle OAD = \angle GBC$, and the triangles OAD and GBC are congruent (sas). Since $GB = GC = R$, we have $OA = OD = R$. The quadrangle $EFBG$ is a rhombus. So $EF \parallel GB \parallel OA$. Moreover, $EF = OA = R$, which means that $AFEO$ is a parallelogram. But this implies $OE = AF = R$. So A, D, and E all are on the circle of radius R centered at O.

87.3. *Let f be a strictly increasing function defined in the set of natural numbers satisfying the conditions $f(2) = a > 2$ and $f(mn) = f(m)f(n)$ for all natural numbers m and n. Determine the smallest possible value of a.*

Solution. Since $f(n) = n^2$ is a function satisfying the conditions of the problem, the smallest posiible a is at most 4. Assume $a = 3$. It is easy to prove by induction that $f(n^k) = f(n)^k$ for all $k \geq 1$. So, taking into account that f is strictly increasing, we get

$$\begin{aligned} f(3)^4 = f(3^4) = f(81) > f(64) = f(2^6) = f(2)^6 \\ = 3^6 = 27^2 > 25^2 = 5^4 \end{aligned}$$

as well as

$$\begin{aligned} f(3)^8 = f(3^8) = f(6561) < f(8192) \\ = f(2^{13}) = f(2)^{13} = 3^{13} < 6^8. \end{aligned}$$

So we arrive at $5 < f(3) < 6$. But this is not possible, since $f(3)$ is an integer. So $a = 4$.

87.4. *Let a, b, and c be positive real numbers. Prove:*

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \leq \frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}.$$

Solution. The arithmetic-geometric inequality yields

$$3 = 3\sqrt[3]{\frac{a^2}{b^2} \cdot \frac{b^2}{c^2} \cdot \frac{c^2}{a^2}} \leq \frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2},$$

or

$$\sqrt{3} \leq \sqrt{\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}}. \tag{1}$$

On the other hand, the Cauchy–Schwarz inequality implies

$$\begin{aligned}\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \le & \sqrt{1^2+1^2+1^2}\sqrt{\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}} \\ =&\sqrt{3}\sqrt{\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}}.\end{aligned} \tag{2}$$

We arrive at the inequality of the problem by combining (1) and (2).

88.1. *The positive integer n has the following property: if the three last digits of n are removed, the number $\sqrt[3]{n}$ remains. Find n.*

Solution. If $x = \sqrt[3]{n}$, and y, $0 \le y < 1000$, is the number formed by the three last digits of n, we have

$$x^3 = 1000x + y.$$

So $x^3 \ge 1000x$, $x^2 > 1000$, and $x > 31$. On the other hand, $x^3 < 1000x + 1000$, or $x(x^2 - 1000) < 1000$. The left hand side of this inequality is an increasing function of x, and $x = 33$ does not satisfy the inequality. So $x < 33$. Since x is an integer, $x = 32$ and $n = 32^3 = 32768$.

88.2. *Let a, b, and c be non-zero real numbers and let $a \ge b \ge c$. Prove the inequality*

$$\frac{a^3-c^3}{3} \ge abc\left(\frac{a-b}{c}+\frac{b-c}{a}\right).$$

When does equality hold?

Solution. Since $c-b \le 0 \le a-b$, we have $(a-b)^3 \ge (c-b)^3$, or

$$a^3 - 3a^2b + 3ab^2 - b^3 \ge c^3 - 3bc^2 + 3b^2c - b^3.$$

On simplifying this, we immediately have

$$\frac{1}{3}(a^3-c^3) \ge a^2b - ab^2 + b^2c - bc^2 = abc\left(\frac{a-b}{c}+\frac{b-c}{a}\right).$$

A sufficient condition for equality is $a = c$. If $a > c$, then $(a-b)^3 > (c-b)^3$, which makes the proved inequality a strict one. So $a = c$ is a necessary condition for equality, too.

88.3. *Two concentric spheres have radii r and R, $r < R$. We try to select points A, B and C on the surface of the larger sphere such that all sides of the triangle ABC would be tangent to the surface of the smaller sphere. Show that the points can be selected if and only if $R \le 2r$.*

Solution. Assume A, B, and C lie on the surface Γ of a sphere of radius R and center O, and AB, BC, and CA touch the surface γ of a sphere of radius r and center O. The circumscribed and inscribed circles of ABC then are intersections of the plane ABC with Γ and γ, respectively. The centers of these circles both are the foot D of the perpendicular dropped from O to the plane ABC. This point lies both on the angle bisectors of the triangle ABC and on the perpendicular bisectors of its sides. So these lines are the same, which means that the triangle ABC is equilateral, and the center of the circles is the common point of intersection of the medians of ABC. This again implies that the radii of the two circles are $2r_1$ and r_1 for some real number r_1. Let $OD = d$. Then $2r_1 = \sqrt{R^2 - d^2}$ and $r_1 = \sqrt{r^2 - d^2}$. Squaring, we get $R^2 - d^2 = 4r^2 - 4d^2$, $4r^2 - R^2 = 3d^2 \ge 0$, and $2r \ge R$.

On the other hand, assume $2r \ge R$. Consider a plane at the distance

$$d = \sqrt{\frac{4r^2 - R^2}{3}}$$

from the common center of the two spheres. The plane cuts the surfaces of the spheres along concentric circles of radii

$$r_1 = \sqrt{\frac{R^2 - r^2}{3}}, \quad R_1 = 2\sqrt{\frac{R^2 - r^2}{3}}.$$

The points A, B, and C can now be chosen on the latter circle in such a way that ABC is equilateral.

88.4. *Let* m_n *be the smallest value of the function*

$$f_n(x) = \sum_{k=0}^{2n} x^k.$$

Show that $m_n \to \frac{1}{2}$, *as* $n \to \infty$.

Solution. For $n > 1$,

$$\begin{aligned} f_n(x) &= 1 + x + x^2 + \cdots \\ &= 1 + x(1 + x^2 + x^4 + \cdots) + x^2(1 + x^2 + x^4 \cdots) \\ &= 1 + x(1+x)\sum_{k=0}^{n-1} x^{2k}. \end{aligned}$$

From this we see that $f_n(x) \geq 1$, for $x \leq -1$ and $x \geq 0$. Consequently, f_n attains its minimum value in the interval $(-1, 0)$. On this interval

$$f_n(x) = \frac{1 - x^{2n+1}}{1 - x} > \frac{1}{1 - x} > \frac{1}{2}.$$

So $m_n \geq \dfrac{1}{2}$. But

$$m_n \leq f_n\left(-1 + \frac{1}{\sqrt{n}}\right) = \frac{1}{2 - \dfrac{1}{\sqrt{n}}} + \frac{\left(1 - \dfrac{1}{\sqrt{n}}\right)^{2n+1}}{2 - \dfrac{1}{\sqrt{n}}}.$$

As $n \to \infty$, the first term on the right hand side tends to the limit $\dfrac{1}{2}$. In the second term, the factor

$$\left(1 - \frac{1}{\sqrt{n}}\right)^{2n} = \left(\left(1 - \frac{1}{\sqrt{n}}\right)^{\sqrt{n}}\right)^{2\sqrt{n}}$$

of the nominator tehds to zero, because

$$\lim_{k\to\infty}\left(1-\frac{1}{k}\right)^k = e^{-1} < 1.$$

So $\lim_{n\to\infty} m_n = \dfrac{1}{2}$.

89.1 *Find a polynomial P of lowest possible degree such that*

(a) *P has integer coefficients,*
(b) *all roots of P are integers,*
(c) *$P(0) = -1$,*
(d) *$P(3) = 128$.*

Solution. Let P be of degree n, and let b_1, b_2, ..., b_m be its zeroes. Then

$$P(x) = a(x-b_1)^{r_1}(x-b_2)^{r_2}\cdots(x-b_m)^{r_m},$$

where r_1, r_2, ..., $r_m \ge 1$, and a is an integer. Because $P(0) = -1$, we have $ab_1^{r_1}b_2^{r_2}\cdots b_m^{r_m}(-1)^n = -1$. This can only happen, if $|a| = 1$ and $|b_j| = 1$ for all $j = 1, 2, \ldots, m$. So

$$P(x) = a(x-1)^p(x+1)^{n-p}$$

for some p, and $P(3) = a\cdot 2^p 2^{2n-2p} = 128 = 2^7$. So $2n-p = 7$. Because $p \ge 0$ and n are integers, the smallest possible n, for which this condition can be true is 4. If $n = 4$, then $p = 1$, $a = 1$. – The polynomial $P(x) = (x-1)(x+1)^3$ clearly satisfies the conditions of the problem.

89.2. *Three sides of a tetrahedron are right-angled triangles having the right angle at their common vertex. The areas of these sides are A, B, and C. Find the total surface area of the tetrahedron.*

Solution 1. Let $PQRS$ be the tetrahedron of the problem and let S be the vertex common to the three sides which are right-angled triangles. Let the areas of PQS, QRS, and RPS be A, B, and C, respectively. Denote the area of QRS by X. If SS' is the altitude from S (onto PQR) and $\angle RSS' = \alpha$, $\angle PSS' = \beta$, $\angle QSS' = \gamma$, the rectangular parallelepiped with SS' as a diameter, gives by double use of the Pythagorean theorem

$$\begin{aligned} SS'^2 &= (SS' \cos\alpha)^2 + (SS' \sin\alpha)^2 \\ &= (SS' \cos\alpha)^2 + (SS' \cos\beta)^2 + (SS' \cos\gamma)^2, \end{aligned}$$

or

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{1}$$

(a well-known formula). The magnitude of the dihedral angle between two planes equals the angle between the normals of the planes. So α, β, and γ are the magnitudes of the dihedral angles between PQR and PQS, QRS, and RPS, respectively. Looking at the projections of PQR onto the three other sides of $PQRS$, we get $A = X\cos\alpha$, $B = X\cos\beta$, and $C = X\cos\gamma$. But (1) now yields $X^2 = A^2 + B^2 + C^2$. The total area of $PQRS$ then equals $A + B + C + \sqrt{A^2 + B^2 + C^2}$.

Solution 2. Use the symbols introduced in the first solution. Align the coordinate axes so that $\overrightarrow{SP} = a\,\overrightarrow{i}$, $\overrightarrow{SQ} = b\,\overrightarrow{j}$, and $\overrightarrow{CR} = c\,\overrightarrow{k}$. The $2A = ab$, $2B = bc$, and $2C = ac$. By the well-known formula for the area of a triangle, we get

$$\begin{aligned} 2X = |\overrightarrow{PQ} \times \overrightarrow{PR}| &= |(b\,\overrightarrow{j} - a\,\overrightarrow{i}) \times (c\,\overrightarrow{k} - a\,\overrightarrow{i})| \\ = |bc\,\overrightarrow{i} + ba\,\overrightarrow{k} + ac\,\overrightarrow{j}| &= 2\sqrt{(bc)^2 + (ba)^2 + (ac)^2} \\ &= 2\sqrt{B^2 + A^2 + c^2}. \end{aligned}$$

So $X = \sqrt{B + A + C}$, and we have $A+B+C+\sqrt{B + A + C}$ as the total area.

89.3. *Let S be the set of all points t in the closed interval $[-1, 1]$ such that for the sequence x_0, x_1, x_2, ... defined by the equations $x_0 = t$, $x_{n+1} = 2x_n^2 - 1$, there exists a positive integer N such that $x_n = 1$ for all $n \geq N$. Show that the set S has infinitely many elements.*

Solution. All numbers in the sequence $\{x_n\}$ lie in the interval $[-1, 1]$. For each n we can pick an α_n such that $x_n = \cos\alpha_n$. If $x_n = \cos\alpha_n$, then $x_{n+1} = 2\cos^2\alpha_n - 1 = \cos(2\alpha_n)$. The nuber α_{n+1} can be chosen as $2\alpha_n$, and by induction, α_n can be chosen as $2^n\alpha_0$. Now $x_n = 1$ if and only if $\alpha_n = 2k\pi$ for some integer k. Take $S' = \{\cos(2^{-m}\pi) | m \in \mathbb{N}\}$. Since every $\alpha_0 = 2^{-m}\pi$ multiplied by a sufficiently large power of 2 is a multiple of 2π, it follows from what was said above that $S' \subset S$. Since S' is infinite, so is S.

89.4 *For which positive integers n is the following statement true: if a_1, a_2, ..., a_n are positive integers, $a_k \leq n$ for all k and $\sum_{k=1}^{n} a_k = 2n$, then it is always possible to choose a_{i_1}, a_{i_2}, ..., a_{i_j} in such a way that the indices i_1, i_2, ..., i_j are different numbers, and $\sum_{k=1}^{j} a_{i_k} = n$?*

Solution. The claim is not true for odd n. A counterexample is provided by $a_1 = a_2 = \cdots = a_n = 2$. We prove by induction that the claim is true for all even $n = 2k$. If $k = 1$, then $a_1 + a_2 = 4$ and $1 \leq a_1$, $a_2 \leq 2$, so necessarily $a_1 = a_2 = 2$. A choice satisfying the condition of the problem is a_1. Now assume that the claim holds for any $2k - 2$ integers with sum $4k - 4$. Let a_1, a_2, ..., a_{2k} be positive integers $\leq 2k$ with sum $4k$. If one of the numbers is $2k$, the case is clear: this number alone can form the required subset. So we may assume that all the numbers are $\leq 2k - 1$. If there are at least two 2's among the numbers, we apply our induction hypothesis to the $2k - 2$ numbers which are left when two 2's are removed. the sum of these numbers is $4k - 4$, so among them there is a subcollection with sum

$2k-2$. Adding one 2 to the collection raises the sum to $2k$. As the next case we assume that there are no 2's among the numbers. Then there must be some 1's among them. Assume there are x 1's among the numbers. Then $2k-x$ of the numbers are ≥ 3. So $x+3(2k-x) \leq 4k$ or $k \leq x$. Now $4k-x$ is between $2k$ and $3k$, and it is and it is the sum of more than one of the numbers in the collection, and these numbers are at least 3 and at most $2k-1$. It follows that we can find numbers ≥ 3 in the collection with sum between k and $2k$. Adding a sufficient number of 1's to this collection we obtain the sum $2k$. We still have the case in which there is exactly one 2 in the collection. Again, denoting the number of 1's by x, we obtain $x+2+3(2k-x-1) \leq 4k$, which implies $2k-1 \leq 2x$. Because x is an integer, we have $k \leq x$. The rest of the proof goes as in the previous case.

90.1. *Let m, n, and p be odd positive integers. Prove that the number*

$$\sum_{k=1}^{(n-1)^p} k^m$$

is divisible by n.

Solution. Since n is odd, the sum has an even number of terms. So we can write it as

$$\sum_{k=1}^{\frac{1}{2}(n-1)^p} \left(k^m + ((n-1)^p - k + 1)^m\right). \tag{1}$$

Because m is odd, each term in the sum has $k+(n-1)^p-k+1 = (n-1)^p+1$ as a factor. As p is odd, too, $(n-1)^p+1 = (n-1)^p+1^p$ has $(n-1)+1 = n$ as a factor. So each term in the sum (1) is divisible by n, and so is the sum.

90.2. *Let a_1, a_2, …, a_n be real numbers. Prove*

$$\sqrt[3]{a_1^3 + a_2^3 + \ldots + a_n^3} \leq \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}. \tag{1}$$

When does equality hold in (1)*?*

Solution. If $0 \le x \le 1$, then $x^{3/2} \le x$, and equality holds if and only if $x = 0$ or $x = 1$. – The inequality is true as an equality, if all the a_k's are zeroes. Assume that at least one of the numbers a_k is non-zero. Set

$$x_k = \frac{a_k^2}{\sum_{j=1}^n a_j^2}.$$

Then $0 \le x_k \le 1$, and by the remark above,

$$\sum_{k=1}^n \left(\frac{a_k^2}{\sum_{j=1}^n a_j^2} \right)^{3/2} \le \sum_{k=1}^n \frac{a_k^2}{\sum_{j=1}^n a_j^2} = 1.$$

So

$$\sum_{k=1}^n a_k^3 \le \left(\sum_{j_1}^n a_j^2 \right)^{3/2},$$

which is what was supposed to be proved. For equality, exactly on x_k has to be one and the rest have to be zeroes, which is equivalent to having exactly one of the a_k's positive and the rest zeroes.

90.3. *Let ABC be a triangle and let P be an interior point of ABC. We assume that a line l, which passes through P, but not through A, intersects AB and AC (or their extensions over B or C) at Q and R, respectively. Find l such that the perimeter of the triangle AQR is as small as possible.*

Solution. (See Figure 2.) Let

$$s = \frac{1}{2}(AR + RQ + QA).$$

Let $\mathcal{C}$ be the excircle of AQR tangent to QR, i.e. the circle tangent to QR and the extensions of AR and AQ. Denote

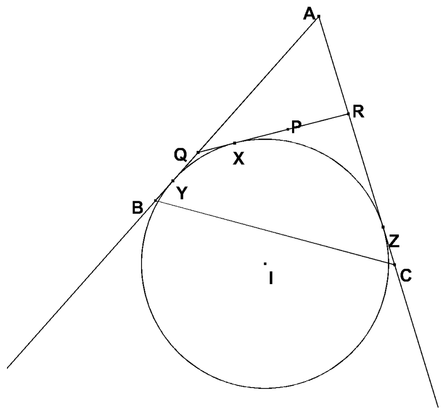


Figure 2.

the center of $\mathcal{C}$ by I and the measure of $\angle QAR$ by α. I is on the bisector of $\angle QAR$. Hence $\angle QAI = \angle IAR = \dfrac{1}{2}\alpha$. Let $\mathcal{C}$ touch RQ, the extension of AQ, and the extension of AR at X, Y, and Z, respectively. Clearly

$$AQ + QX = AY = AZ = AR + RX,$$

so

$$AZ = AI \cos \frac{1}{2}\alpha = s.$$

Hence s and the perimeter of AQR is smallest, when AI is smallest. If $P \neq X$, it is possible to turn the line through P to push $\mathcal{C}$ deeper into the angle BAC. So the minumum for AI is achieved precisely as $X = P$. To construct minimal triangle, we have to draw a circle touching the half lines AB and AC and passing through P. This is accomplished by first drawing an arbitrary circle touching the half lines, and then performing a suitable homothetic transformation of the circle to make it pass through P.

90.4. *It is possible to perform three operations f, g, and h for positive integers: $f(n) = 10n$, $g(n) = 10n + 4$, and $h(2n) = n$; in other words, one may write 0 or 4 in the end of the number and one may divide an even number by 2. Prove: every positive integer can be constructed starting from 4 and performing a finite number of the operations f, g, and h in some order.*

Solution. All odd numbers n are of the form $h(2n)$. All we need is to show that every even number can be obtained fron 4 by using the operations f, g, and h. To this end, we show that a suitably chosen sequence of inverse operations $F = f^{-1}$, $G = g^{-1}$, and $H = h^{-1}$ produces a smaller even number or the number 4 from every positive even integer. The operation F can be applied to numbers ending in a zero, the operation G can be applied to numbers ending in 4, and $H(n) = 2n$. We obtain

$$\begin{gathered} H(F(10n)) = 2n, \\ G(H(10n+2)) = 2n, \quad n \geq 1, \\ H(2) = 4, \\ H(G(10n+4)) = 2n, \\ G(H(H(10n+6))) = 4n+2, \\ G(H(H(H(10n+8)))) = 8n+6. \end{gathered}$$

After a finite number of these steps, we arrive at 4.

91.1. *Determine the last two digits of the number*

$$2^5 + 2^{5^2} + 2^{5^3} + \cdots + 2^{5^{1991}},$$

written in decimal notation.

Solution. We first show that all numbers 2^{5^k} are of the form $100p + 32$. This can be shown by induction. The case

$k = 1$ is clear ($2^5 = 32$). Assume $2^{5^k} = 100p + 32$. Then, by the binomial formula,

$$2^{5^{k+1}} = \left(2^{5^k}\right)^5 = (100p + 32)^5 = 100q + 32^5$$

and

$$\begin{aligned}(30+2)^5 &= 30^5+5\cdot30^4\cdot2+10\cdot30^3\cdot4+10\cdot30^2\cdot8+5\cdot30\cdot16+32\\ &= 100r + 32.\end{aligned}$$

So the last two digits of the sum in the problem are the same as the last digits of the number $1991 \cdot 32$, or 12.

91.2. *In the trapezium $ABCD$ the sides AB and CD are parallel, and E is a fixed point on the side AB. Determine the point F on the side CD so that the area of the intersection of the triangles ABF and CDE is as large as possible.*

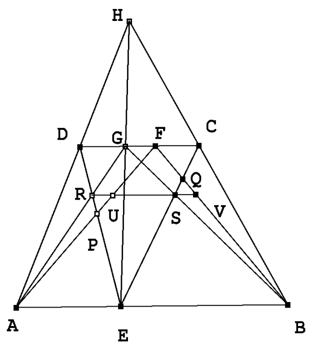


Figure 3.

Solution 1. (See Figure 3.) We assume $CD < AB$. Let AD and BC intersect at H and EH and DC at G. Let DE

intersect AF at P and FB intersect EC at Q. Denote the area of a figure $\mathcal{F}$ by $|\mathcal{F}|$. Since $|ABF|$ does not depend on the choice of F on DC, $|EQFP|$ is maximized when $|AEP|+|EBQ|$ is minimized. We claim that this takes place when $F = G$. Let R and S be the points of intersection of the trapezia $AEGD$ and $EBCG$, respectively. Then $RS \| AB$. (To see this, consider the pairs AER and GDR; EBS and CGS of similar triangles. The ratios of their altitudes are $AE : DG$ and $EB : GC$, respectively. But both ratios are equal to $EG : HG$. As the sum of the ratios in both pairs is the altitude of $ABCD$, the altitudes of, say AER and EBS are equal, which inplies the claim.) Denote the points where RS intersects FA and FB by U and V, respectively. Then $|AUR| = |BVS|$. (RU and SV are the same fraction of GF, and both triangles have the same altitude.) Assume that F lies between G and C. Then

$$|APE| + |EBQ| > |APE| + |EBS| + |BSV|$$
$$= |APE| + |EBS| + |AUR| > |APE| + |EBS| + |APR|$$
$$= |ARE| + |EBS|.$$

A similar inequality can be established, when F is between G and D. So the choice $F = G$ minimizes $|AEP| + |EBQ|$ and maximizes $|EQFP|$. – Proofs in the cases $AB = CD$ and $AB < CD$ go along similar lines.

Solution 2. We again minimize $|AEP|+|EBQ|$. Set $AB = a$, $CD = b$, $AE = c$, $DF = x$, and denote the altitude of $ABCD$ by h and the altitudes of AEP and EBQ by h_1 and h_2, respectively. Since AEP and FDP are similar, as well as EBQ and CFQ, we have

$$\frac{c}{x} = \frac{h_1}{h - h_1}, \quad \text{and} \quad \frac{a-c}{b-x} = \frac{h_2}{h-h_2}.$$

Solving from these, we obtain

$$h_1 = \frac{ch}{x+c}, \quad h_2 = \frac{(a-c)h}{a+b-c-x}.$$

As $h_1c + h_2(a-c)$ is double the area to be minimized, we seek the minimum of

$$f(x) = \frac{c^2}{x+c} + \frac{(a-c)^2}{2a-c-x}.$$

The necessary minimum condition $f'(x) = 0$ means

$$\frac{c^2}{(x+c)^2} = \frac{(a-c)^2}{(a+b-c-x)^2}.$$

Solving this, we obtain $x = \dfrac{bc}{a}$, and since the left hand side of the equation has a decreasing and the right hand side an increasing function of x in the relevant interval $0 \le x \le b$, we see that $x = c$ is the only root of $f')x) = 0$, and we also note that $f'(x)$ is increasing. So $f(x)$ has a global minimum at $x = \dfrac{bc}{a}$. This means that, in terms of the notation of the first solution, $F = G$ is the solution of the problem.

91.3. *Show that*

$$\frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} < \frac{2}{3}$$

for all $n \ge 2$.

Solution. Since

$$\frac{1}{j^2} < \frac{1}{j(j-1)} = \frac{1}{j-1} - \frac{1}{j},$$

we have

$$\sum_{j=k}^{n} \frac{1}{j^2} < \left(\frac{1}{k-1} - \frac{1}{k}\right) + \left(\frac{1}{k} - \frac{1}{k+1}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right)$$
$$= \frac{1}{k-1} - \frac{1}{n} < \frac{1}{k-1}.$$

From this we obtain for $k = 6$

$$\frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} < \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{5} = \frac{2389}{3600} < \frac{2}{3}.$$

91.4. *Let $f(x)$ be a polynomial with integer coefficients. We assume that there exists a positive integer k and k consecutive integers n, $n+1$, ..., $n+k-1$ so that none of the numbers $f(n)$, $f(n+1)$, ..., $f(n+k-1)$ is divisible by k. Show that the zeroes of $f(x)$ are not integers.*

Solution. Let $f(x) = a_0x^d + a_1x^{d-1} + \cdots + a_d$. Assume that f has a zero m which is an integer. Then $f(x) = (x-m)g(x)$, where g is a polynomial. If $g(x) = b_0x^{d-1} + b_1x^{d-2} + \cdots + b_{d-1}$, then $a_0 = b_0$, and $a_k = b_k - mb_{k-1}$, $1 \le k \le d-1$. So b_0 is an integer, and by induction all b_k's are integers. Because $f(j)$ is not divisible by k for k consequtive values of j, no one of the k consequtive integers $j-m$, $j = n$, $n+1$, ..., $n+k-1$, is divisible by k. But this is a contradiction, since exactly one of k consequtive integers is divisible by k. So f cannot have an integral zero.

92.1. *Determine all real numbers $x > 1$, $y > 1$, and $z > 1$, satisfying the equation*

$$x + y + z + \frac{3}{x-1} + \frac{3}{y-1} + \frac{3}{z-1} = 2\left(\sqrt{x+2} + \sqrt{y+2} + \sqrt{z+2}\right).$$

Solution. Consider the function f,

$$f(t) = t + \frac{3}{t-1} - 2\sqrt{t+2},$$

defined for $t > 1$. The equation of the problem can be written as

$$f(x) + f(y) + f(z) = 0.$$

We reformulate the formula for f:

$$\begin{aligned} f(t) &= \frac{1}{t-1}\left(t^2 - t + 3 - 2(t-1)\sqrt{t+2}\right) \\ &= \frac{1}{t-1}\left(t^2 - 2t + 1 + \left(\sqrt{t+2}\right)^2 - 2(t-1)\sqrt{t+2}\right) \\ &= \frac{1}{t-1}\left(t - 1 - \sqrt{t+2}\right)^2. \end{aligned}$$

So $f(t) \geq 0$, and $f(t) = 0$ for $t > 1$ only if

$$t - 1 = \sqrt{t+2}$$

or

$$t^2 - 3t - 1 = 0.$$

The only t satisfying this condition is

$$t = \frac{3 + \sqrt{13}}{2}.$$

So the only solution to the equation in the problem is given by

$$x = y = z = \frac{3 + \sqrt{13}}{2}.$$

92.2. *Let $n > 1$ be an integer and let a_1, a_2, …, a_n be n different integers. Show that the polynomial*

$$f(x) = (x - a_1)(x - a_2) \cdots (x - a_n) - 1$$

is not divisible by any polynomial with integer coefficients and of degree greater than zero but less than n and such that the highest power of x has coefficient 1.

Solution. Suppose $g(x)$ is a polynomial of degree m, where $1 \le m < n$, with integer coefficients and leading coefficient 1, such that

$$f(x) = g(x)h(x),$$

whre $h(x)$ is a polynomial. Let

$$\begin{aligned} g(x) &= x^m + b_{m-1}x^{m-1} + \cdots + b_1x + b_0, \\ h(x) &= x^{n-m} + c_{n-m-1}x^{n-m-1} + \cdots + c_1x + c_0. \end{aligned}$$

We show that the coefficients of $h(x)$ are integers. If they are not, there is a greatest index $j = k$ such that c_k is not an integer. But then the coefficient of f multiplying x^{k+m} – which is an integer – would be $c_k + b_{m-1}c_{k+1} + b_{m-2}c_{k+2} + \ldots b_{k-m}$. All terms except the first one in this sum are integers, so the sum cannot be an integer. A contradiction. So $h(x)$ is a polynomial with integral coefficients. Now

$$f(a_i) = g(a_i)h(a_i) = -1,$$

for $i = 1, 2, \ldots, n$, and $g(a_i)$ and $h(a_i)$ are integers. This is only possible, if $g(a_i) = -f(a_i) = \pm 1$ and $g(a_i) + h(a_i) = 0$ for $i = 1, 2, \ldots, n$. So the polynomial $g(x) + h(x)$ has at least n zeroes. But the degree of $g(x) + h(x)$ is less than n. So $g(x) = -h(x)$ for all x, and $f(x) = -g(x)^2$. This is impossible, however, because $f(x) \to +\infty$, as $x \to +\infty$. This contradiction proves the claim.

92.3 *Prove that among all triangles with inradius 1, the equilateral one has the smallest perimeter.*

Solution. (See Figure 4.) The area T, perimeter p and inradius r satisfy $2T = rp$. (Divide the triangle into three triangles with a common vertex at the incenter of the triangle.) So for a fixed inradius, the triangle with the smallest perimeter is the one which has the smallest area. To prove that the equilateral triangles minimize the area among triangles with a fixed incircle, we utilize three trivial facts, which the reader may prove for his/her enjoyment:

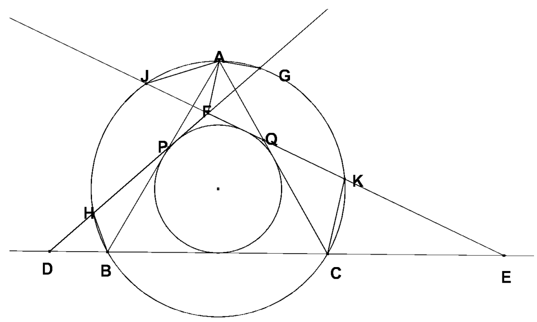


Figure 4.

Lemma 1. If AB and CD are two equal chords of a circle and if they intersect at P, and if D is on the shorter arc AB, then APD and CPB are congruent triangles.

Lemma 2. If $\mathcal{C}_1$ and $\mathcal{C}_2$ concentric circles, then all chords of $\mathcal{C}_1$ which are tangent to $\mathcal{C}_2$ are equal.

Lemma 3. Given a circle $\mathcal{C}$, the set of points P such that the tangents to $\mathcal{C}$ through P meet at a fixed angle, is a circle concentric to $\mathcal{C}$.

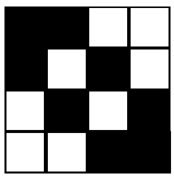
Now consider an equilateral triangle ABC with incircle $\mathcal{C}_1$ and circumcircle $\mathcal{C}_2$. Let DEF be another triangle with incircle $\mathcal{C}_1$. If DEF is not equilateral, it either has two angles $< 60°$ and one angle $> 60°$, two angles $> 60°$ and one angle $< 60°$, or one angle < 60, one $= 60°$, and one $> 60°$. In the first case, using Lemma 3 and its immediate consequences, we may rotate the triangles and rename the vertices so that F is inside $\mathcal{C}_2$ and D and E are outside it. Let DF intersect $\mathcal{C}_2$ at G and H, let EF intersect $\mathcal{C}_2$ at K and J (J on the shorter arc HG), and let AB and HG intersect at P, and AC and JK at Q. Since A is on different sides of HG and JK than B and C, respectively, A must be on the shorter arc JG. By Lemma 1, BPH and APG are congruent and JQA and QCK are congruent. We compute, denoting the area of a figure $\mathcal{F}$ by $|\mathcal{F}|$:

$$\begin{aligned}
|FDE| &= |ABC| + |DBP| - |PFA| + |QCE| - |AFQ| \\
&> |ABC| + |PHB| - |PFA| + |CKQ| - |AFG| \\
&> |ABC| + |PHB| - |PGA| + |CKQ| - |QAJ| = |ABC|.
\end{aligned}$$

The two other cases can be treated analogously.

92.4. *Peter has many squares of equal side. Some of the squares are black, some are white. Peter wants to assemble a big square, with side equal to n sides of the small squares, so that the big square has no rectangle formed by the small squares such that all the squares in the vertices of the rectangle are of equal colour. How big a square is Peter able to assemble?*

Solution. We show that Peter only can make a 4×4 square. The construction is possible, if $n = 4$:



Now consider the case $n = 5$. We may assume that at least 13 of the 25 squares are black. If five black squares are on one horizontal row, the remaining eight ones are distributed on the other four rows. At least one row has two black squres. A rectangle with all corners black is created. Next assume that one row has four black squares. Of the remaing 9 squares, at least three are one row. At least two of these three have to be columns having the assumed four black squares. If no row has more than four black squares, there have to be at least three rows with exactly three black squares. Denote these rows by A, B, and C. Let us call the columns in which the black squares on row A lie *black columns*, and the other two columns *white columns*. If either row B or row C has at least two black squares which are on black columns, a rectancle with black corners arises. If both rows B and C have only one black square on the black columns, then both of them have two black squares on the two white columns, and they make the black corners of a rectangle. So Peter cannot make a 5×5 square in the way he wishes.

93.1. *Let F be an increasing real function defined for all x, $0 \le x \le 1$, satisfying the conditions*

(i) $\qquad F\left(\dfrac{x}{3}\right) = \dfrac{F(x)}{2}$,

(ii) $\qquad F(1-x) = 1 - F(x)$.

Determine $F\left(\dfrac{173}{1993}\right)$ *and* $F\left(\dfrac{1}{13}\right)$.

Solution. Condition (i) implies $F(0) = \dfrac{1}{2}F(0)$, so $F(0) = 0$. Because of condition (ii), $F(1) = 1 - F(0) = 1$. Also $F\left(\dfrac{1}{3}\right) = \dfrac{1}{2}$ and $F\left(\dfrac{2}{3}\right) = 1 - F\left(\dfrac{1}{3}\right) = \dfrac{1}{2}$. Since F is an increasing function, this is possible only if $F(x) = \dfrac{1}{2}$ for all $x \in \left[\dfrac{1}{3}, \dfrac{2}{3}\right]$. To determine the first of the required values of F, we use (i) and (ii) to transform the argument into the middle third of $[0, 1]$:

$$\begin{aligned} F\left(\frac{173}{1993}\right) &= \frac{1}{2}F\left(\frac{519}{1993}\right) = \frac{1}{4}F\left(\frac{1557}{1993}\right) \\ &= \frac{1}{4}\left(1 - F\left(\frac{436}{1993}\right)\right) = \frac{1}{4}\left(1 - \frac{1}{2}F\left(\frac{1308}{1993}\right)\right) \\ &= \frac{1}{4}\left(1 - \frac{1}{4}\right) = \frac{3}{16}. \end{aligned}$$

To find the second value of F, we use (i) and (ii) to form an equation fron which the value can be solved. Indeed,

$$F\left(\frac{1}{13}\right) = 1 - F\left(\frac{12}{13}\right) = 1 - 2F\left(\frac{4}{13}\right)$$
$$= 1 - 2\left(1 - F\left(\frac{9}{13}\right)\right) = 2F\left(\frac{9}{13}\right) - 1 = 4F\left(\frac{3}{13}\right) - 1$$
$$= 8F\left(\frac{1}{13}\right) - 1.$$

From this we solve

$$F\left(\frac{1}{13}\right) = \frac{1}{7}.$$

93.2. *A hexagon is inscribed in a circle of radius r. Two of the sides of the hexagon have length 1, two have length 2 and two have length 3. Show that r satisfies the equation*

$$2r^3 - 7r - 3 = 0.$$

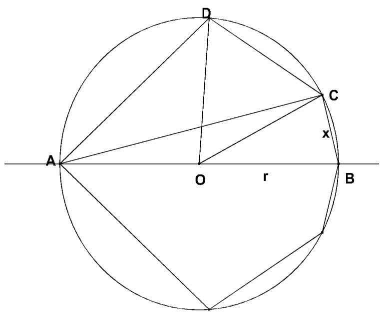


Figure 5.

Solution. (See Figure 5.) We join the vertices of the hexagon to the center O of its circumcircle. We denote by α the central angles corresponding the chords of length 1, by β those corresponding the chords of length 2, and by γ those corresponding the chords of length 3. Clearly $\alpha+\beta+\gamma = 180^\circ$. We can move three chords of mutually different length so that they follow each other on the circumference. We thus obtain a quadrilateral $ABCD$ where $AB = 2r$ is a diameter of the circle, $BC = 1$, $CD = 2$, and $DA = 3$. Then $\angle COB = \alpha$ and $\angle CAB = \dfrac{\alpha}{2}$. Then $\angle ABC = 90^\circ - \dfrac{\alpha}{2}$, and, as $ABCD$ is an inscribed quafdrilateral, $\angle CDA = 90^\circ + \dfrac{\alpha}{2}$. Set $AC = x$. From triangles ABC and ACD we obtain $x^2 + 1 = 4r^2$ and

$$x^2 = 4 + 9 - 2\cdot 2\cdot 3\cos\left(90^\circ + \frac{\alpha}{2}\right) = 13 + 12\sin\left(\frac{\alpha}{2}\right).$$

From triangle ABC,

$$\sin\left(\frac{\alpha}{2}\right) = \frac{1}{2r}.$$

We put this in the expression for x^2 to obtain

$$4r^2 = x^2 + 1 = 14 + 12\cdot\frac{1}{2r}$$

which is equivalent to

$$2r^3 - 7r - 3 = 0.$$

93.3. *Find all solutions of the system of equations*

$$\begin{cases} s(x) + s(y) = x \\ x + y + s(z) = z \\ s(x) + s(y) + s(z) = y - 4, \end{cases}$$

where x, y, and z are positive integers, and $s(x)$, $s(y)$, and $s(z)$ are the numbers of digits in the decimal representations of x, y, and z, respectively.

Solution. The first equation implies $x \geq 2$ and the first and third equation together imply

$$s(z) = y - x - 4. \qquad (1)$$

So $y \geq x + 5 \geq 7$. From (1) and the second equation we obtain $z = 2y - 4$. Translated to the values of s, these equation imply $s(x) \leq s(2y) \leq s(y)+1$ and $s(x) \leq s(y)$. We insert these inequalitien in the last equation of the problem to obtain $y-4 \leq 3s(y)+1$ or $y \leq 3s(y)+5$. Since $10^{s(y)-1} \leq y$, the only possible values of $s(y)$ are 1 and 2. If $s(y) = 1$, then $7 \leq y \leq 3 + 5 = 8$. If $y = 7$, x must be 2 and $z = 2 \cdot 7 - 4 = 10$. But this does not fit in the second equation: $2 + 7 + 2 \neq 10$. If $y = 8$, then $z = 12$, $x = 2$. The triple (2, 8, 12) satisfies all the equations of the problem. If $s(y) = 2$, then $y \leq 6 + 5 = 11$. The only possibilities are $y = 10$ and $y = 11$. If $y = 10$, then $z = 16$ and $x \leq 5$. The equation $s(x) + s(y) + s(z) = y - 4 = 6$ is not satisfied. If $y = 11$, then $z = 18$ and $x \leq 6$. Again, the third equation is not satisfied. So $x = 2$, $y = 8$, and $z = 12$ is the only solution.

93.4. *Denote by $T(n)$ the sum of the digits of the decimal representation of a positive integer n.*

a) *Find an integer N, for which $T(k \cdot N)$ is even for all k, $1 \leq k \leq 1992$, but $T(1993 \cdot N)$ is odd.*

b) *Show that no positive integer N exists such that $T(k \cdot N)$ is even for all positive integers k.*

Solution. a) If s has n decimal digits and $m = 10^{n+r}s + s$, then $T(km)$ is even at least as long as $ks < 10^{n+r}$, because all non-zero digits appear in pairs in km. Choose $N = 5018300050183$ or $s = 50183$, $n = 5$, $r = 3$. Now $1992 \cdot s = 99964536 < 10^8$, so $T(kN)$ is even for all $k \leq 1992$. But $1993 \cdot s = 100014719$, $1993 \cdot N = 10001472000014719$, and $T(1993 \cdot N) = 39$ is odd.

b) Assume that N is a positive integer for which $T(kN)$ is even for all k. Consider the case $N = 2m$ first. Then $T(km) = T(10km) = T(5kN)$. As $T(5kN)$ is even for every k, then so is $T(km)$. Repeating the argument suffiently many times we arrive at an odd N, such that $T(kN)$ is even for all k. Assume now $N = 10r + 5$. Then $T(k(2r + 1)) = T(10k(2r + 1)) = T(2kN)$. From this we conclude that the number $\frac{N}{5} = 2r+1$ has the the property we are dealing with. By repeating the argument, we arrive at an odd number N, which does not have 5 as a factor, such that $T(kN)$ is even for all k. Next assume $N = 10r+9$. If N has n digits and the decimal representation of N is $\overline{ax\ldots xb9}$, where the x's can be any digits, then, if $b < 9$, the decimal representation of $10^{n-1}N + N$ is $\overline{ax\ldots x(b+1)(a-1)x\ldots xb9}$. This implies $T(10^{n-2}N + N) = 2T(N) - 9$, which is an odd number. If the second last digit b of N is 9, then $11N$ has 89 as its two last digits, and again we see that N has a multiple kN with $T(kn)$ odd. Finally, if the last digit of N is 1, the last digit of $9N$ is 9, if the last digit of N is 3, the last digit of $3N$ is 9, and if the last digit of N is 7, the last digit of $7N$ is 9. All these cases thus can be reduced to the cases already treated. So all odd numbers have multiples with an odd sum of digits, and the proof is complete.

94.1. *Let O be an interior point in the equilateral triangle ABC, of side length a. The lines AO, BO, and CO intersect the sides of the triangle in the points A_1, B_1, and C_1. Show that*

$$|OA_1| + |OB_1| + |OC_1| < a.$$

Solution. Let H_A, H_B, and H_C be the orthogonal projections of O on BC, CA, and AB, respectively. Because

$60° < \angle OA_1B < 120°$,

$$|OH_A| = |OA_1|\sin(\angle OA_1B) > |OA_1|\frac{\sqrt{3}}{2}.$$

In the same way,

$$|OH_B| > |OB_1|\frac{\sqrt{3}}{2} \quad \text{and} \quad |OH_C| > |OC_1|\frac{\sqrt{3}}{2}.$$

The area of ABC is $a^2\frac{\sqrt{3}}{4}$ but also $\frac{a}{2}(OH_A+OH_B+OH_C)$ (as the sum of the areas of the three triangles with common vertex O which together comprise ABC). So

$$|OH_A| + |OH_B| + |OH_C| = a\frac{\sqrt{3}}{2},$$

and the claim follows at once.

94.2. *We call a finite plane set S consisting of points with integer coefficients* a two-neighbour set*, if for each point (p, q) of S exactly two of the points $(p+1, q)$, $(p, q+1)$, $(p-1, q)$, $(p, q-1)$ belong to S. For which integers n there exists a two-neighbour set which contains exactly n points?*

Solution. The points $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$ clearly form a two-neighbour set (which we abbreviate as 2NS). For every even number $n = 2k \geq 8$, the set $S = \{(0, 0), \ldots, (k-2, 0), (k-2, 1), (k-2, 2), \ldots, (0, 2), (0, 1)\}$ is a 2NS. We show that there is no 2NS with n elements for other values n.

Assume that S is a 2NS and S has n points. We join every point in S to two of its neighbours by a unit line segment. The ensuing figures are closed polygonal lines, since an end-point of such a line would have only one neighbour. The polygons contains altogether n segments (from each point, two segments emanate, and counting the emanating segments means that the segments will be counted twice.) In each of the polygons, the number of segments is even. When walking around such a polygon one has to take equally many steps to the left as to the right, and equally many up and down. Also, $n \neq 2$.

What remains is to show is that $n \neq 6$. We may assume $(0, 0) \in S$. For reasons of symmetry, essentially two possibilities exist: a) $(-1, 0) \in S$ and $(1, 0) \in S$, or b) $(1, 0) \in S$ and $(0, 1) \in S$. In case a), $(0, 1) \notin S$ and $(0, -1) \notin S$. Because the points $(-1, 0)$, $(0, 0)$, and $(1, 0)$ of S belong to a closed polygonal line, this line has to wind around either $(0, 1)$ or $(0, -1)$. In both cases, the polygon has at least 8 segments. In case b) $(1, 1) \notin S$ (because otherwise S would generate two polygons, a square an one with two segments). Also $(-1, 0) \notin S$, and $(0, -1) \notin S$. The polygon which contains $(1, 0)$, $(0, 0)$, and $(0, 1)$ thus either winds around the point $(1, 1)$, in which case it has at least 8 segments, or it turns around the points $(-1, 0)$ and $(0, -1)$, in which case it has at least 10 segments. So $n = 6$ always leads to a contradiction.

94.3. *A piece of paper is the square $ABCD$. We fold it by placing the vertex D on the point H of the side BC. We assume that AD moves onto the segment GH and that HG intersects AB at E. Prove that the perimeter of the triangle EBH is one half of the perimeter of the square.*

Solution. (See Figure 6.) The fold gives rise to an isosceles trapezium $ADHG$. Because of symmetry, the distance of the vertex D from the side GH equals the distance of the vertex H from side AD; the latter distance is the side length a of the square. The line GH thus is tangent to the circle with center D and radius a. The lines AB and BC are tangent to the same circle. If the point common to GH and the circle is F, then $AE = EF$ and $FH = HC$. This implies

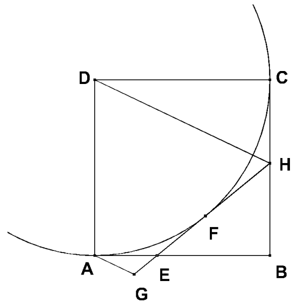


Figure 6.

$AB+BC = AE+EB+BH+HC = EF+EB+BH+HF = EH+EB+BH$, which is equivalent to what we were asked to prove.

94.4. *Determine all positive integers* $n < 200$*, such that* $n^2 + (n+1)^2$ *is the square of an integer.*

Solution. We determine the integral solutions of

$$n^2 + (n+1)^2 = (n+p)^2, \quad p \geq 2.$$

The root formula for quadratic equations yields

$$n = p - 1 + \sqrt{2p(p-1)} \geq 2(p-1).$$

Because $n < 200$, we have $p \leq 100$. Moreover, the number $2p(p-1)$ has to be the square of an integer. If p is odd, p and $2(p-1)$ have no common factors. Consequently, both p and $2(p-1)$ have to be squares. The only possible candidates are $p = 9$, $p = 25$, $p = 49$, $p = 81$. The respective numbers $2(p-1)$ are 16, 48, 96, and 160. Of these, only 16 is a square. We thus have one solution $n = 8 + \sqrt{2 \cdot 9 \cdot 8} = 20$, $20^2 + 21^2 = 841 = 29^2$. If p is even, the numbers $2p$ and $p-1$ have no factors in common, so both are squares. Possible candidates for $2p$ are 4, 16, 36, 64, 100, 144, and 196. The corresponding values of $p-1$ are 1, 7, 31, 49, 71, 97. We obtain two more solutions: $n = 1+2 = 3$, $3^2+4^2 = 5^2$, and $n = 49+70 = 119$, $119^2+120^2 = 169^2$.

95.1. *Let AB be a diameter of a circle with centre O. We choose a point C on the circumference of the circle such that OC and AB are perpendicular to each other. Let P be an arbitrary point on the (smaller) arc BC and let the lines CP and AB meet at Q. We choose R on AP so that RQ and AB are perpendicular to each other. Show that $|BQ| = |QR|$.*

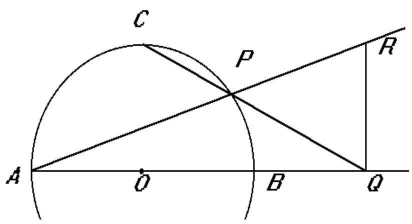


Figure 7.

Solution 1. (See Figure 7.) Draw PB. By the Theorem of Thales, $\angle RPB = \angle APB = 90°$. So P and Q both lie on the circle with diameter RB. Because $\angle AOC = 90°$, $\angle RPQ = \angle CPA = 45°$. Then $\angle RBQ = 45°$, too, and RBQ is an isosceles right triangle, or $|BQ| = |QR|$.

Solution 2. Set $O = (0, 0)$, $A = (-1, 0)$, $B = (1, 0)$, $C = (0, 1)$, and $P = (t, u)$, where $t > 0$, $u > 0$, and $t^2 + u^2 = 1$. The equation of line CP is $y - 1 = \dfrac{u-1}{t}x$. So $Q = \left(\dfrac{t}{1-u}, 0\right)$ and $|BQ| = \dfrac{t}{1-u} - 1 = \dfrac{t+u-1}{1-u}$. On the other hand, the equation of line AP is $y = \dfrac{u}{t+1}(x+1)$.

The y coordinate of R and also $|QR|$ is $\dfrac{u}{t+1}\left(\dfrac{t}{1-u}+1\right) = \dfrac{ut+u-u^2}{(t+1)(1-u)} = \dfrac{ut+u-1+t^2}{(t+1)(1-u)} = \dfrac{u+t-1}{1-u}$. The claim has been proved.

95.2. *Messages are coded using sequences consisting of zeroes and ones only. Only sequences with at most two consecutive ones or zeroes are allowed. (For instance the sequence 011001 is allowed, but 011101 is not.) Determine the number of sequences consisting of exactly 12 numbers.*

Solution 1. Let S_n be the set of acceptable sequences consisting of $2n$ digits. We partition S_n in subsets A_n, B_n, C_n, and D_n, on the basis of the two last digits of the sequence. Sequences ending in 00 are in A_n, those ending in 01 are in B_n, those ending in 10 are in C_n, and those ending in 11 are in D_n. Denote by x_n, a_n, b_n, c_n, and d_n the number of elements in S_n, A_n, B_n, C_n, and D_n. We compute x_6. Because $S_1 = \{00, 01, 10, 11\}$, $x_1 = 4$ and $a_1 = b_1 = c_1 = d_1 = 1$. Every element of A_{n+1} can be obtained in a unique manner from an element of B_n or D_n by adjoining 00 to the end. So $a_{n+1} = b_n + d_n$. The elements of B_{n+1} are similarly obtained from elements of B_n, C_n, and D_n by adjoining 01 to the end. So $b_{n+1} = b_n + c_n + d_n$. In a similar manner we obtain the recursion formulas $c_{n+1} = a_n + b_n + c_n$ and $d_{n+1} = a_n + c_n$. So $a_{n+1} + d_{n+1} = (b_n + d_n) + (a_n + c_n) = x_n$ and $x_{n+1} = 2a_n + 3b_n + 3c_n + 2d_n = 3x_n - (a_n + b_n) = 3x_n - x_{n-1}$. Starting from the initial values $a_1 = b_1 = c_1 = d_1 = 1$, we obtain $a_2 = d_2 = 2$, $b_2 = c_2 = 3$, and $x_2 = 10$. So $x_3 = 3x_2 - x_1 = 3 \cdot 10 - 4 = 26$, $x_4 = 3 \cdot 26 - 10 = 68$, $x_5 = 3 \cdot 68 - 26 = 178$, and $x_6 = 3 \cdot 178 - 68 = 466$.

Solution 2. We can attach a sequence of ones and twos to each acceptable sequence by indicating the number of consequtive equal numbers; these one's and twos then add up to the length of the sequence. Interchnaging all ones and zeros in the sequence results in another acceptabe sequence which in turn yields the same sequence of ones and twos. Thus any way of writing 12 as a sum of ones and twos, in a specified order, corresponds to exactly two acceptable sequences of lenghth 12. The number of sums with 12 ones is one, the number of sums with one 2 and 10 ones is $\binom{11}{10}$ etc. The number of acceptable sequences is

$$2 \cdot \sum_{k=0}^{6} \binom{12-k}{2k} = 2 \cdot (1+11+45+84+70+21+1) = 466.$$

95.3. *Let $n \geq 2$ and let x_1, x_2, $\ldots x_n$ be real numbers satisfying $x_1 + x_2 + \ldots + x_n \geq 0$ and $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$. Let $M = \max\{x_1, x_2, \ldots, x_n\}$. Show that*

$$M \geq \frac{1}{\sqrt{n(n-1)}}. \tag{1}$$

When does equality hold in (1)*?*

Solution. Denote by I the set of indices i for which $x_i \geq 0$, and by J the set of indices j for which $x_j < 0$. Let us assume $M < \dfrac{1}{\sqrt{n(n-1)}}$. Then $I \neq \{1, 2, \ldots, n\}$, since otherwise we would have $|x_i| = x_i \leq \dfrac{1}{\sqrt{n(n-1)}}$ for every i, and $\sum_{i=1}^n x_i^2 < \dfrac{1}{n-1} \leq 1$. So $\sum_{i \in I} x_i^2 < (n-1) \cdot \dfrac{1}{n(n-1)} = \dfrac{1}{n}$, and $\sum_{i \in I} x_i < (n-1)\dfrac{1}{\sqrt{n(n-1)}} = \sqrt{\dfrac{n-1}{n}}$. Because

$$0 \leq \sum_{i=1}^{n} x_i = \sum_{i \in I} x_i - \sum_{i \in J} |x_i|,$$

we must have $\sum_{i\in J}|x_i| \le \sum_{i\in I} x_i < \sqrt{\frac{n-1}{n}}$ and $\sum_{i\in J} x_i^2 \le \left(\sum_{i\in J}|x_i|\right)^2 < \frac{n-1}{n}$. But then

$$\sum_{i=1}^{n} x_i^2 = \sum_{i\in I} x_i^2 + \sum_{i\in J} x_i^2 < \frac{1}{n} + \frac{n-1}{n} = 1,$$

and we have a contradiction. – To see that equality $M = \frac{1}{\sqrt{n(n-1)}}$ is possible, we choose $x_i = \frac{1}{\sqrt{n(n-1)}}$, $i = 1$, 2, …, $n-1$, and $x_n = -\sqrt{\frac{n-1}{n}}$. Now

$$\sum_{i=1}^{n} x_i = (n-1)\frac{1}{\sqrt{n(n-1)}} - \sqrt{\frac{n-1}{n}} = 0$$

and

$$\sum_{i=1}^{n} x_i^2 = (n-1)\cdot\frac{1}{n(n-1)} + \frac{n-1}{n} = 1.$$

We still have to show that equality can be obtained only in this case. Assume $x_i = \frac{1}{\sqrt{n(n-1)}}$, for $i = 1, \ldots, p$, $x_i \ge 0$, for $i \le q$, and $x_i < 0$, kun $q+1 \le i \le n$. As before we get

$$\sum_{i=1}^{q} x_i \le \frac{q}{\sqrt{n(n-1)}}, \quad \sum_{i=q+1}^{n} |x_i| \le \frac{q}{\sqrt{n(n-1)}},$$

and

$$\sum_{i=q+1}^{n} x_i^2 \le \frac{q^2}{n(n-1)},$$

so

$$\sum_{i=1}^{n} x_i^2 \le \frac{q+q^2}{n^2-n}.$$

It is easy to see that $q^2+q < n^2+n$, for $n \ge 2$ and $q \le n-2$, but $(n-1)^2+(n-1) = n^2-n$. Consequently, a necessary condition for $M = \dfrac{1}{\sqrt{n(n-1)}}$ is that the sequence only has one negative member. But if among the positive members there is at least one smaller than M we have

$$\sum_{i=1}^{n} < \frac{q+q^2}{n(n-1)},$$

so the conditions of the problem are not satisfied. So there is equality if and only if $n-1$ of the numbers x_i equal $\dfrac{1}{\sqrt{n(n-1)}}$, and the last one is $\dfrac{1-n}{\sqrt{n(n-1)}}$.

95.4. *Show that there exist infinitely many mutually non-congruent triangles T, satisfying*

(i) *The side lengths of T are consecutive integers.*

(ii) *The area of T is an integer.*

Solution. Let $n \ge 3$, and let $n-1$, n, $n+1$ be the side lengths of the triangle. The semiperimeter of the triangle then equals on $\dfrac{3n}{2}$. By Heron's formula, the area of the triangle is

$$\begin{aligned} T &= \sqrt{\frac{3n}{2} \cdot \left(\frac{3n}{2} - n + 1\right)\left(\frac{3n}{2} - n\right)\left(\frac{3n}{2} - n - 1\right)} \\ &= \frac{n}{2}\sqrt{\frac{3}{4}(n^2-4)}. \end{aligned}$$

If $n = 4$, then $T = 6$. So at least one triangle of the kind required exists. We prove that we always can form new triangles of the required kind from ones already known to exist. Let n be even, $n \geq 4$, and let $\frac{3}{4}(n^2-4)$ be a square number. Set $m = n^2-2$. Then $m > n$, m is even, and $m^2-4 = (m+2)(m-2) = n^2(n^2-4)$. So $\frac{3}{4}(m^2-4) = n^2 \cdot \frac{3}{4}(n^2-4)$ is a square number. Also, $T = \frac{m}{2}\sqrt{\frac{3}{4}(m^2-4)}$ is an integer. The argument is complete.

96.1. *Show that there exists an integer divisible by 1996 such that the sum of the its decimal digits is 1996.*

Solution. The sum of the digits of 1996 is 25 and the sum of the digits of $2\cdot 1996 = 3992$ is 23. Because $1996 = 78\cdot 25+46$, the number obtained by writing 78 1996's and two 3992 in succession satisfies the condition of the problem. – As $3 \cdot 1996 = 5998$, the sum of the digits of 5988 is 30, and $1996 = 65 \cdot 30 + 46$, the number $39923992\underbrace{5988\ldots5988}_{65\ \text{times}}$ also can be be given as an answer, indeed a better one, as it is much smaller than the first suggestion.

96.2. *Determine all real numbers x, such that*

$$x^n + x^{-n}$$

is an integer for all integers n.

Solution. Set $f_n(x) = x^n + x^{-n}$. $f_n(0)$ is not defined for any n, so we must have $x \neq 0$. Since $f_0(x) = 2$ for all $x \neq 0$, we have to find out those $x \neq 0$ for which $f_n(x)$ is an integer foe every $n > 0$. We note that

$$x^n + x^{-n} = (x + x^{-1})(x^{n-1} + x^{1-n}) - (x^{n-2} + x^{2-n}).$$

From this we obtain by induction that x^n+x^{-n} is an integer for all $n > 1$ as soon as $x + x^{-1}$ is an integer. So x has to satisfy

$$x + x^{-1} = m,$$

where m is an integer. The roots of this quadratic equation are

$$x = \frac{m}{2} \pm \sqrt{\frac{m^2}{4} - 1},$$

and they are real, if $m \neq -1, 0, 1$.

96.3. *The circle whose diameter is the altitude dropped from the vertex A of the triangle ABC intersects the sides AB and AC at D and E, respectively ($A \neq D$, $A \neq E$). Show that the circumcentre of ABC lies on the altitude dropped from the vertex A of the triangle ADE, or on its extension.*

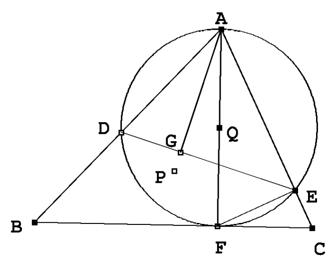


Figure 8.

Solution. (See Figure 8.) Let AF be the altitude of ABC. We may assume that $\angle ACB$ is sharp. From the right triangles ACF and AFE we obtain $\angle AFE = \angle ACF$. $\angle ADE$ and $\angle AFE$ subtend the same arc, so they are equal. Thus $\angle ACB = \angle ADE$, and the triangles ABC and AED are similar. Denote by P and Q the circumcenters of ABC and AED, respectively. Then $\angle BAP = \angle EAQ$. If AG is the altitude of AED, then $\angle DAG = \angle CAF$. But this implies $\angle BAP = \angle DAG$, which means that P is on the altitude AG.

96.4. *The real-valued function f is defined for positive integers, and the positive integer a satisfies*

$$f(a) = f(1995), \quad f(a+1) = f(1996), \quad f(a+2) = f(1997)$$

$$f(n+a) = \frac{f(n)-1}{f(n)+1} \quad \textit{for all positive integers } n.$$

(i) *Show that $f(n+4a) = f(n)$ for all positive integers n.*

(ii) *Determine the smallest possible a.*

Solution. To prove (i), we the formula $f(n+a) = \dfrac{f(n)-1}{f(n)+1}$ repeatedly:

$$f(n+2a) = f((n+a)+a) = \frac{\dfrac{f(n)-1}{f(n)+1} - 1}{\dfrac{f(n)-1}{f(n)+1} + 1} = -\frac{1}{f(n)},$$

$$f(n+4a) = f((n+2a)+2a) = -\frac{1}{-\dfrac{1}{f(n)}} = f(n).$$

(ii) If $a = 1$, then $f(1) = f(a) = f(1995) = f(3+498\cdot 4a) = f(3) = f(1+2a) = -\dfrac{1}{f(1)}$. This clearly is not possible, since $f(1)$ and $\dfrac{1}{f(1)}$ have equal sign. So $a \neq 1$.

If $a = 2$, we obtain $f(2) = f(a) = f(1995) = f(3+249\cdot 4a) = f(3) = f(a+1) = f(1996) = f(4+249\cdot 4a) = f(4) = f(2+a) = \dfrac{f(2)-1}{f(2)+1}$, or $f(2)^2 + f(2) = f(2) - 1$. This quadratic equation in $f(2)$ has no real solutions. So $a \neq 2$.

If $a = 3$, we try to construct f by choosing $f(1)$, $f(2)$, and $f(3)$ arbitrarily and by computing the other values of f by the recursion formula $f(n+3) = \dfrac{f(n)-1}{f(n)+1}$. We have to check that f defined in this way satisfies the conditions of the problem.

The condition

$$f(n+a) = f(n+3) = \frac{f(n)-1}{f(n)+1}$$

is valid because of the construction. Further, by (i),

$$f(n+12) = f(n+4a) = f(n),$$

which implies

$$f(a) = f(3) = f(3+166\cdot 12) = f(1995),$$

$$f(a+1) = f(4) = f(4+166\cdot 12) = f(1996),$$
$$f(a+2) = f(5) = f(5+166\cdot 12) = f(1997)$$

as required.

We remark that the choice $f(n) = -1$ makes $f(n+3)$ undefined, the choice $f(n) = 0$ makes $f(n+3) = -1$ and $f(n+6)$ is undefined, and $f(n) = 1$ makes $f(n+3) = 0$ so $f(n+9)$ is undefined. In the choice of $f(1)$, $f(2)$, and $f(3)$ we have to avoid -1, 0, 1.

In conclusion, we see that $a = 3$ is the smallest possible value for a.

97.1. *Let A be a set of seven positive numbers. Determine the maximal number of triples (x, y, z) of elements of A satisfying $x < y$ and $x + y = z$.*

Solution. Let $0 < a_1 < a_2 < \ldots < a_7$ be the elements of the set A. If (a_i, a_j, a_k) is a triple of the kind required in the problem, then $a_i < a_j < a_i + a_j = a_k$. There are at most $k-1$ pairs (a_i, a_j) such that $a_i + a_j = a_k$. The number of pairs satisfying $a_i < a_j$ is at most $\left\lfloor \frac{k-1}{2} \right\rfloor$. The total number of pairs is at most

$$\sum_{k=3}^{7} \left\lfloor \frac{k-1}{2} \right\rfloor = 1 + 1 + 2 + 2 + 3 = 9.$$

The value 9 can be reached, if $A = \{1, 2, \ldots, 7\}$. In this case the triples (1, 2, 3), (1, 3, 4), (1, 4, 5), (1, 5, 6), (1, 6, 7), (2, 3, 5), (2, 4, 6), (2, 5, 7), and (3, 4, 7) satisfy the conditions of the problem.

97.2. *Let $ABCD$ be a convex quadrilateral. We assume that there exists a point P inside the quadrilateral such that the areas of the triangles ABP, BCP, CDP, and DAP are equal. Show that at least one of the diagonals of the quadrilateral bisects the other diagonal.*

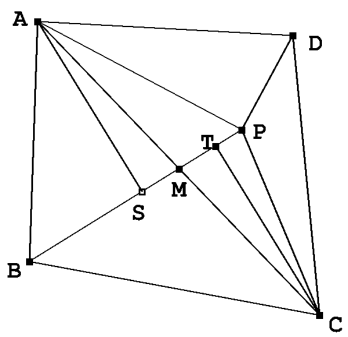


Figure 9.

Solution. (See Figure 9.) We first assume that P does not lie on the diagonal AC and the line BP meets the diagonal

AC at M. Let S and T be the feet of the perpendiculars from A and C on the line BP. The triangles APB and CBP have equal area. Thus $AS = CT$. If $S \neq T$, then the right trianges ASM and CTM are congruent, and $AM = CM$. If, on the other hand, $S = T$, the $AC \perp PB$ and $S = M = T$, and again $AM = CM$. In both cases M is the midpoint of the diagonal AC. We prove exactly in the same way that the line DP meets AC at the midpoint of AC, i.e. at M. So B, M, and P, and also D, M, and P are collinear. So M is on the line DB, which means that BD divides the diagonal AC in two equal parts.

We then assume that P lies on the diagonal AC. Then P is the midpoint of AC. If P is not on the diagonal BD, we argue as before that AC divides BD in two equal parts. If P lies also on the diagonal BD, it has to be the common midpoint of the diagonals.

97.3. *Let A, B, C, and D be four different points in the plane. Three of the line segments AB, AC, AD, BC, BD, and CD have length a. The other three have length b, where $b > a$. Determine all possible values of the quotient $\frac{b}{a}$.*

Solution. If the three segments of length a share a common endpoint, say A, then the other three points are on a circle of radius a, centered at A, and they are the vertices of an equilateral triangle of side length b. But this means that A is the center of the triangle BCD, and

$$\frac{b}{a} = \frac{b}{\frac{2}{3}\frac{\sqrt{3}}{2}b} = \sqrt{3}.$$

Assume then that of the segments emanating from A at least one has lenght a and at least one has length b. We may assume $AB = a$ and $AD = b$. If only one segment of length a would emanate from each of the four poits, then the number of segments of length a would be two, as every segment is counted twice when we count the emanating segments. So we may assume that AC has length a, too. If $BC = a$, then ABC would be an equilateral triangle, and the distance of D from each of its vertices would be b. This is not possible, since $b > a$. So $BC = b$. Of the segments CD and BD one has length a. We may assume $DC = a$. The segments DC and AB are either on one side of thye line AC or on opposite sides of it. In the latter case, $ABCD$ is a parallelogram with a pair of sides of length a and a pair of sides of length b, and its diagonals have lengths a and b. This is not possible, due to the fact that the sum of the squares of the diagonals of the parallelogram, $a^2 + b^2$, would be equal to the sum of the squares of its sides, i.e. $2a^2 + 2b^2$. This means that we may assume that $BACD$ is a convex quadrilateral. Let $\angle ABC = \alpha$ and $\angle ADB = \beta$. From isosceles triangles we obtain for instance $\angle CBD = \beta$, and from the triangle ABD in particular $2\alpha + 2\beta + \beta = \pi$ as well as $\angle CDA = \alpha$, $\angle DCB = \frac{1}{2}(\pi - \beta)$, $\angle CAD = \alpha$. The triangle ADC thus yields $\alpha + \alpha + \alpha + \frac{1}{2}(\pi - \beta) = \pi$. From this we solve $\alpha = \frac{1}{5}\pi = 36°$. The sine theorem applied to ABC gives

$$\frac{b}{a} = \frac{\sin 108°}{\sin 36°} = \frac{\sin 72°}{\sin 36°} = 2\cos 36° = \frac{\sqrt{5}+1}{2}.$$

(In fact, a is the side of a regular pentagon, and b is its diagonal.) – Another way of finding the ratio $\dfrac{b}{a}$ is to consider the trapezium $CDBA$, with $CD \| AB$; if E is the orthogonal projection of B on the segment CD, then $CE = b - \dfrac{1}{2}(b-a) = \dfrac{1}{2}(b+a)$. The right triangles BCE and DCE yield $CE^2 = b^2 - \left(\dfrac{b+a}{2}\right)^2 = a^2 - \left(\dfrac{b-a}{2}\right)^2$, which can be written as $b^2 - ab - a^2 = 0$. From this we solve $\frac{b}{a} = \frac{\sqrt{5}+1}{2}$.

97.4. *Let f be a function defined in the set $\{0, 1, 2, \ldots\}$ of non-negative integers, satisfying $f(2x) = 2f(x)$, $f(4x+1) = 4f(x) + 3$, and $f(4x - 1) = 2f(2x - 1) - 1$. Show that f is an injection, i.e. if $f(x) = f(y)$, then $x = y$.*

Solution. If x is even, then $f(x)$ is even, and if x is odd, then $f(x)$ is odd. Moreover, if $x \equiv 1 \bmod 4$, then $f(x) \equiv 3 \bmod 4$, and if $x \equiv 3 \bmod 4$, then $f(x) \equiv 1 \bmod 4$. Clearly $f(0) = 0$, $f(1) = 3$, $f(2) = 6$, and $f(3) = 5$. So at least f restricted to the set $\{0, 1, 2, 3\}$ ia an injection. We prove that $f(x) = f(y) \Longrightarrow x = y$, for $x, y < k$ implies $f(x) = f(y) \Longrightarrow x = y$, for $x, y < 2k$. So assume x and y are smaller than $2k$ and $f(x) = f(y)$. If $f(x)$ is even, then $x = 2t$, $y = 2u$, and $2f(t) = 2f(u)$. As t and u are less than k, we have $t = u$, and $x = y$. Assume $f(x) \equiv 1 \bmod 4$. Then $x \equiv 3 \bmod 4$; $x = 4u - 1$, and $f(x) = 2f(2u - 1) - 1$. Also $y = 4t - 1$ and $f(y) = 2f(2t - 1) - 1$. Moreover, $2u - 1 < \frac{1}{2}(4u - 1) < k$ and $2t - 1 < k$, so $2u - 1 = 2t - 1$, $u = t$, and $x = y$. If, finally, $f(x) \equiv 3 \bmod 4$, then $x = 4u + 1$, $y = 4t + 1$, $u < k$, $t < k$, $4f(u) + 3 = 4f(t) + 3$, $u = t$, and $x = y$. Since for all x and y there is an n such that the larger one of the numbers x and y is $< 2^n \cdot 3$, the induction argument above shows that $f(x) = f(y) \Rightarrow x = y$.

98.1. *Determine all functions f defined in the set of rational numbers and taking their values in the same set such that the equation $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ holds for all rational numbers x and y.*

Solution. Insert $x = y = 0$ in the equation to obtain $2f(0) = 4f(0)$, which implies $f(0) = 0$. Setting $x = 0$, one obtains $f(y) + f(-y) = 2f(y)$ of $f(-y) = f(y)$. Then assume $y = nx$, where n is a positive integer. We obtain

$$f((n+1)x) = 2f(x) + 2f(nx) - f((n-1)x).$$

In particular, $f(2x) = 2f(x) + 2f(x) - f(0) = 4f(x)$ and $f(3x) = 2f(x) + 2f(2x) - f(x) = 9f(x)$. We prove $f(nx) = n^2 f(x)$ for all positive integers n. This is true for $n = 1$. Assume $f(kx) = k^2 f(x)$ for $k \le n$. Then

$$\begin{aligned} f((n+1)x) &= 2f(x) + 2f(nx) - f((n-1)x) \\ &= (2 + 2n^2 - (n-1)^2)f(x) = (n+1)^2 f(x), \end{aligned}$$

and we are done. If $x = 1/q$, where q is a positive integer, $f(1) = f(qx) = q^2 f(x)$. So $f(1/q) = f(1)/q^2$. This again implies $f(p/q) = p^2 f(1/q) = (p/q)^2 f(1)$. We have shown that there is a rational number $a = f(1)$ such that $f(x) = ax^2$ for all positive rational numbers x. But since f is an even function, $f(x) = ax^2$ for all rational x. We still have to check that for every rational a, $f(x) = ax^2$ satisfies the conditions of the problem. In fact, if $f(x) = ax^2$, then $f(x+y) + f(x-y) = a(x+y)^2 + a(x-y)^2 = 2ax^2 + 2ay^2 = 2f(x) + 2f(y)$. So the required functions are all functions $f(x) = ax^2$ where a is any rational number.

98.2. *Let C_1 and C_2 be two circles intersecting at A and B. Let S and T be the centres of C_1 and C_2, respectively. Let P be a point on the segment AB such that $|AP| \ne |BP|$ and $P \ne A$, $P \ne B$. We draw a line perpendicular to SP through P and denote by C and D the points at which this line intersects C_1. We likewise draw a line perpendicular to TP through P and denote by E and F the points at which this line intersects C_2. Show that C, D, E, and F are the vertices of a rectangle.*

Solution. (See Figure 10.) The power of the point P with respect to the circles C_1 and C_2 is $PA \cdot PB = PC \cdot PD =$

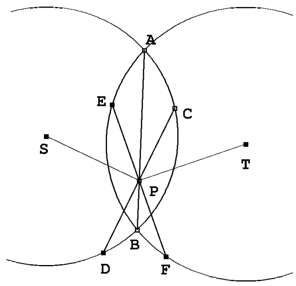


Figure 10.

$PE \cdot PF$. Since SP is perpendicular to the chord CD, P has to be the midpoint of CD. So $PC = PD$. In a similar manner, we obtain $PE = PF$. Alltogether $PC = PD = PE = PF = \sqrt{PA \cdot PB}$. Consequently the points C, D, E, and F all lie on a circle withe center P, and CD and EF as diameters. By Thales' theorem, the angles $\angle ECF$, $\angle CFD$ etc. are right angles. So $CDEF$ is a rectangle.

98.3. *(a) For which positive numbers n does there exist a sequence x_1, x_2, …, x_n, which contains each of the numbers 1, 2, …, n exactly once and for which $x_1 + x_2 + \cdots + x_k$ is divisible by k for each $k = 1, 2, \ldots, n$?*

(b) Does there exist an infinite sequence x_1, x_2, x_3, …, which contains every positive integer exactly once and such that $x_1 + x_2 + \cdots + x_k$ is divisible by k for every positive integer k?

Solution. (a) We assume that x_1, …, x_n is the sequence required in the problem. Then $x_1+x_2+\cdots+x_n = \dfrac{n(n+1)}{2}$. This sum should be divisible by n. If n is odd, this is possible, since $\dfrac{(n+1)}{2}$ is an integer. If, on the other hand, $n =$

$2m$, then $\dfrac{n(n+1)}{2} = m(2m+1) = 2m^2 + m \equiv m \bmod 2m$. So even n's are ruled out. Assume $n = 2m+1 > 1$. We require that $n-1 = 2m$ divides evenly the number $x_1 + \cdots + x_{n-1}$. Since $x_1 + \cdots + x_{n-1} = (m+1)(2m+1) - x_n \equiv m + 1 - x_n \bmod 2m$, and $1 \le x_n \le n$, we must have $x_n = m+1$. We also require that $n-2 = 2m-1$ divides evenly the number $x_1 + \cdots + x_{n-2}$. Now $x_1 + \cdots + x_{n-2} = (m+1)(2m+1) - x_n - x_{n-1} \equiv m+1-x_{n-1} \bmod (2m-1)$ and $-m \le m+1-x_{n-1} \le m$, we have $x_{n-1} = m+1 \bmod (2m-1)$. If $n > 3$ or $m \ge 1$, we must have $x_{n-1} = m+1 = x_n$, which is not allowed. So the only possibilities are $n = 1$ or $n = 3$. If $n = 1$, $x_1 = 1$ is a possible sequence. If $n = 3$, we must have $x_3 = 2$. x_1 and x_2 are 1 and 3 in any order.

(b) Let $x_1 = 1$. We define the sequence by a recursion formula. Assume that $x_1, x_2, \ldots, x_{n-1}$ have been chosen and that the sum of these numbers is A. Let m be the smallest integer not yet chosen into the sequence. If x_{n+1} is chosen to be m, there will be two restrictions on x_n:

$$A + x_n \equiv 0 \bmod n \quad \text{and} \quad A + x_n + m \equiv 0 \bmod n+1.$$

Since n and $n+1$ are relatively prime, there exists, by the Chinese Remainder Theorem, a y such that $y \equiv -A \bmod n$ and $y \equiv -A - m \bmod n+1$. If one adds a suitably large multiple of $n(n+1)$ to y, one obtains a number not yet in the sequence. So the sequence always can be extended by two numbers, and eventually every positive integer will be included.

98.4. *Let n be a positive integer. Count the number of numbers $k \in \{0, 1, 2, \ldots, n\}$ such that $\binom{n}{k}$ is odd. Show that this number is a power of two, i.e. of the form 2^p for some nonnegative integer p.*

Solution. The number of odd binomial coefficients $\binom{n}{k}$ equals the number of ones on the n:th line of the Pascal Triangle mod 2:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 0 & & 1 & & & & \\
 & & & 1 & & 1 & & 1 & & 1 & & & \\
 & & 1 & & 0 & & 0 & & 0 & & 1 & & \\
 & 1 & & 1 & & 0 & & 0 & & 1 & & 1 & \\
1 & & 0 & & 1 & & 0 & & 1 & & 0 & & 1
\end{array}$$

(We count the lines so that the uppermost line is line 0). We notice that line 1 has two copies of line 0, lines 2 and 3 contain two copies of lines 1 and 2, etc.

The fundamental property $\binom{n+1}{p} = \binom{n}{p-1} + \binom{n}{p}$ of the Pascal Triangle implies that if all numbers on line k are $\equiv 1 \bmod 2$, then on line $k+1$ exactly the first and last numbers are $\equiv 1 \bmod 2$. If, say on line k exactly the first and last numbers are $\equiv 1 \bmod 2$, then the lines k, $k+1$, $\ldots$, $2k-1$ are formed by two copies of lines $0, 1, \ldots k-1$, separated by zeroes. As line 0 has number 1 and line 1 is formed by two ones, the lines 2 and three are formed by two copies of lines 0 and 1, etc. By induction we infer that for every k, the line $2^k - 1$ is forned of ones only – it has two copies of line $2^{k-1} - 1$, and the line $0 = 2^0 - 1$ is a one. The line 2^k has ones in the end and zeroes in between. Now let N_n be the number of ones on line $n = 2^k + m$, $m < 2^k$. Then $N_1 = 2$ and $N_n = 2N_m$. So N_n always is a power of two. To be more precise, we show that $N_n = 2^{e(n)}$, where $e(n)$ is the number of ones in the binary representation of n. The formula is true for $n = 0$, as $N_0 = 1 = 2^{e(0)}$. Also, if $m < 2^k$, $e(2^k + m) = e(m) + 1$. On the other hand, if $n = 2^k + m$, $m < 2^k$ then $N_n = 2N_m = 2 \cdot 2^{e(m)} = 2^{e(m)+1} = 2^{e(n)}$.

99.1. *The function f is defined for non-negative integers and satisfies the condition*

$$f(n) = \begin{cases} f(f(n+11)), & \text{if } n \le 1999 \\ n-5, & \text{if } n > 1999. \end{cases}$$

Find all solutions of the equation $f(n) = 1999$.

Solution. If $n \ge 2005$, then $f(n) = n - 5 \ge 2000$, and the equation $f(n) = 1999$ has no solutions. Let $1 \le k \le 4$. Then

$$\begin{aligned} 2000 - k &= f(2005 - k) = f(f(2010 - k)) \\ &= f(1999 - k) = f(f(2004 - k)) = f(1993 - k). \end{aligned}$$

Let $k = 1$. We obtain three solutions $1999 = f(2004) = f(1998) = f(1992)$. Moreover, $1995 = f(2000) = f(f(2005)) = f(1994)$ and $f(1993) = f(f(2004)) = f(1999) = f(f(2010)) = f(2005) = 2000$. So we have shown that $2000 - k = f(1999 - k)$, for $k = 0, 1, 2, 3, 4, 5$, and $2000 - k = f(1993 - k)$ for $k = 0, 1, 2, 3, 4$. We now show by downwards induction that $f(6n + 1 - k) = 2000 - k$ for $n \le 333$ and $0 \le k \le 5$. This has already been proved for $n = 333$ and $n = 332$. We assume that the claim is true for $n = m + 2$ and $n = m + 1$. Then $f(6m + 1 - k) = f(f(6m + 12 - k)) = f(f(6(m + 2) + 1 - (k + 1)) = f(2000 - k - 1) = f(1999 - k) = 2000 - k$ for $k = 0, 1, 2, 3, 4$, and $f(6m + 1 - 5) = f(6m - 4) = f(f(6m + 7)) = f(f(6(m + 1) + 1)) = f(2000) = 1995 = 2000 - 5$. So the claim is true for $n = m$. Summing up, $1999 = 2000 - 1 = f(6n)$, if and only if $n = 1, 2, \ldots, 334$.

99.2. *Consider 7-gons inscribed in a circle such that all sides of the 7-gon are of different length. Determine the maximal number of 120° angles in this kind of a 7-gon.*

Solution. It is easy to give examples of heptagons $ABCDEFG$ inscribed in a circle with all sides unequal and two angles equal to 120°. These angles cannot lie on adjacent vertices of the heptagon. In fact, if $\angle ABC = \angle BCD = 120^\circ$, and arc BC equals b°, then arcs AB and CD both are $120^\circ - b^\circ$ (compute angles in isosceles triangles with center of the circle as the to vertex), and $AB = CD$, contrary to the assumption. So if the heptagon has three angles of 120°, their vertices are, say A, C, and E. Then each of the arcs GAB, BCD, DEF are $360^\circ - 240^\circ = 120^\circ$. The arcs are disjoint, so they cover the whole circumference. The F has to coincide with G, and the heptagon degenerates to a hexagon. There can be at most two 120° angles.

99.3. *The infinite integer plane $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$ consists of all number pairs (x, y), where x and y are integers. Let a and b be non-negative integers. We call any move from a point (x, y) to any of the points $(x \pm a, y \pm b)$ or $(x \pm b, y \pm a)$ a (a, b)-knight move. Determine all numbers a and b, for which it is possible to reach all points of the integer plane from an arbitrary starting point using only (a, b)-knight moves.*

Solution. If the greatest common divisor of a and b is d, only points whose coordinates are multiples of d can be reached by a sequence of (a, b)-knight moves starting from the origin. So $d = 1$ is a necessary condition for the possibility of reaching every point in the integer plane. In any (a, b)-knight move, $x + y$ either stays constant or increases or diminishes by $a + b$. If $a + b$ is even, then all points which can be reached from the origin have an even coordinate sum. So $a+b$ has to be odd. We now show that if $d = 1$ and $a+b$ is odd, then all points can be reached. We may assume $a \geq 1$ and $b \geq 1$, for if $ab = 0$, $d = 1$ is possible only if one of the numbers a, b is 0 and the other one 1. In this case clearly all points can be reached. Since $d = 1$, there exist positive numbers r and s such that either $ra-sb=1$ or $sb-ra=1$. Assume $ra-sb=1$. Make r moves $(x, y) \to (x+a, y+b)$ and r moves $(x, y) \to (x+a, y-b)$ to travel from point (x, y) to point $(x+2ra, y)$. After this, make s moves $(x, y) \to (x-b, a)$ and s moves $(x, y) \to (x-b, -a)$ to arrive at point $(x+2ra-2sb, y) = (x+2, y)$. In a similar manner we construct sequences of moves carrying us from point (x, y) to points $(x-2, y)$, $(x, y+2)$, and $(x, y-2)$. This means that we can reach all points with both coordinates even from the origin. Exactly one of the numbers a and b is odd. We may assume $a = 2k+1$, $b = 2m$. A move $(x, y) \to (x+a, y+b) = (x+1+2k, y+2m)$, followed by k sequences of moves $(x, y) \to (x-2, y)$ and m sequences of moves $(x, y) \to (x, y-2)$ takes us to the point $(x+1, y)$. In a similar manner we reach the points $(x-1, y)$ and $(x, y\pm1)$ from (x, y). So all points can be reached from the origin. – If $sb-ra=1$, the argument is similar.

99.4. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers and $n \geq 1$. Show that*

$$n\left(\frac{1}{a_1}+\cdots+\frac{1}{a_n}\right) \geq \left(\frac{1}{1+a_1}+\cdots+\frac{1}{1+a_n}\right)\left(n+\frac{1}{a_1}+\cdots+\frac{1}{a_n}\right).$$

When does equality hold?

Solution. The inequality of the problem can be written as

$$\frac{1}{1+a_1}+\cdots+\frac{1}{1+a_n} \leq \frac{n\left(\dfrac{1}{a_1}+\cdots+\dfrac{1}{a_n}\right)}{n+\dfrac{1}{a_1}+\cdots+\dfrac{1}{a_n}}.$$

A small manipulation of the right hand side brings the inequality to the equivalent form

$$\frac{1}{\dfrac{1}{a_1^{-1}}+1}+\cdots+\frac{1}{\dfrac{1}{a_n^{-1}}+1} \le \frac{n}{\dfrac{1}{\dfrac{a_1^{-1}+\cdots+a_n^{-1}}{n}}+1}. \qquad (1)$$

Consider the function

$$f(x) = \frac{1}{\dfrac{1}{x}+1} = \frac{x}{1+x}.$$

We see that it is concave, i.e.

$$tf(x) + (1-t)f(y) < f(tx + (1-t)y)$$

for all $t \in (0, 1)$. In fact, the inequality

$$t\frac{x}{1+x} + (1-t)\frac{y}{1+y} < \frac{tx + (1-t)y}{1 + tx + (1-t)y}$$

can be written as

$$t^2(x-y)^2 < t(x-y)^2,$$

and because $0 < t < 1$, it is true. [Another standard way to see this is to compute

$$f'(x) = \frac{1}{(1+x)^2}, \quad f''(x) = -\frac{2}{(1+x)^3} < 0.$$

A function with a positive second derivative is concave.] For any concave function f, the inequality

$$\frac{1}{n}(f(x_1) + f(x_2) + \cdots + f(x_n)) \le f\left(\frac{x_1 + \cdots + x_n}{n}\right)$$

holds, with equality only for $x_1 = x_2 = \ldots = x_n$. So (1) is true, and equality holds only if all a_i's are equal.

00.1. *In how many ways can the number 2000 be written as a sum of three positive, not necessarily different integers? (Sums like* $1+2+3$ *and* $3+1+2$ *etc. are the same.)*

Solution. Since 3 is not a factor of 2000, there has to be at least two different numbers among any three summing up to 2000. Denote by x the number of such sums with three different summands and by y the number of sums with two different summands. Consider 3999 boxes consequtively numbered fron 1 to 3999 such that all boxes labelled by an odd number contain a red ball. Every way to put two blue balls in the even-numbered boxes produces a partition of 2000 in three summands. There are $\binom{1999}{2} = 999 \cdot 1999$ ways to place the blue balls. But htere are $3! = 6$ different placements, which produce the same partition of 2000 into three different summands, and $\frac{3!}{2} = 3$ different placements, which produce the same partition of 2000 into summands two which are equal. Thus $6x+3y = 1999\cdot 999$. But $y = 999$, because the number appering twice in the partition can be any of the numbers $1, 2, \ldots, 999$. This leads to $x = 998\cdot 333$, so $x + y = 1001 \cdot 333 = 333333$.

00.2. *The persons* P_1, P_1, $\ldots$, P_{n-1}, P_n *sit around a table, in this order, and each one of them has a number of coins. In the start,* P_1 *has one coin more than* P_2, P_2 *has one coin more than* P_3, *etc., up to* P_{n-1} *who has one coin more than* P_n. *Now* P_1 *gives one coin to* P_2, *who in turn gives two coins to* P_3 *etc., up to* P_n *who gives* n *coins to* P_1. *Now the process continues in the same way:* P_1 *gives* $n+1$ *coins to* P_2, P_2 *gives* $n+2$ *coins to* P_3; *in this way the transactions go on until someone has not enough coins, i.e. a person no more can give away one coin more than he just received. At the moment when the process comes to an end in this manner, it turns out that there are to neighbours at* *the table such that one of them has exactly five times as many coins as the other. Determine the number of persons and the number of coins circulating around the table.*

Solution. Assume that P_n has m coins in the start. Then P_{n-1} has $m+1$ coins, ... and P_1 has $m+n-1$ coins. In every move a player receives k coins and gives $k+1$ coins away, so her net loss is one coin. After the first round, when P_n has given n coins to P_1, P_n has $m-1$ coins, P_{n-1} has m coins etc., after two rounds P_n has $m-2$ coins, P_{n-1} has $m-1$ coins etc. This can go on during m rounds, after which P_n has no money, P_{n-1} has one coin etc. On round $m+1$ each player still in possession of money can receive and give away coins as before. The penniless P_n can no more give away coins according to the rule. She receives $n(m+1)-1$ coins from P_{n-1}, but is unable to give $n(m+1)$ coins to P_1. So when the game ends, P_{n-1} has no coins and P_1 has $n-2$ coins. The only pair of neighbours such that one has 5 times as many coins as the other can be (P_1, P_n). Because $n-2 < n(m+1)-1$, this would mean $5(n-2) = n(m+1)-1$ or $n(4-m) = 9$. Since $n > 1$, the possibilities are $n = 3$, $m = 1$ or $n = 9$, $m = 3$. Both are indeed possible. In the first case the number of coins is $3+2+1 = 6$, in the second $11+10+\cdots+3 = 63$.

00.3. *In the triangle ABC, the bisector of angle B meets AC at D and the bisector of angle C meets AB at E. The bisectors meet each other at O. Furthermore, $OD = OE$. Prove that either ABC is isosceles or $\angle BAC = 60°$.*

Solution. (See Figure 11.) Consider the triangles AOE and AOD. They have two equal pairs of sides and the angles facing one of these pairs are equal. Then either AOE and AOD are congruent or $\angle AEO = 180° - \angle ADO$. In the first case, $\angle BEO = \angle CDO$, and the triangles EBO and DCO are congruent. Then $AB = AC$, and ABC is isosceles. In the second case, denote the angles of ABC by 2α, 2β,

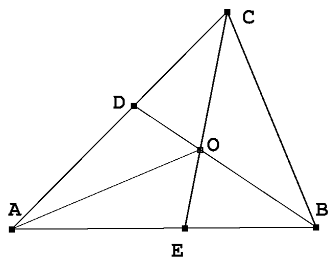


Figure 11.

and 2γ, and the angle AEO by δ. By the theorem on the adjacent angle of an angle of a triangle, $\angle BOE = \angle DOC = \beta + \gamma$, $\delta = 2\beta + \gamma$, and $180° - \delta = \beta + 2\gamma$. Adding these equations yields $3(\beta+\gamma) = 180°$ eli $\beta+\gamma = 60°$. Combining this with $2(\alpha + \beta + \gamma) = 180°$, we obtain $2\alpha = 60°$.

00.4. *The real-valued function f is defined for $0 \le x \le 1$, $f(0) = 0$, $f(1) = 1$, and*

$$\frac{1}{2} \le \frac{f(z) - f(y)}{f(y) - f(x)} \le 2$$

for all $0 \le x < y < z \le 1$ with $z - y = y - x$. Prove that

$$\frac{1}{7} \le f\left(\frac{1}{3}\right) \le \frac{4}{7}.$$

Solution. We set $f\left(\frac{1}{3}\right) = a$ and $f\left(\frac{2}{3}\right) = b$. Applying the inequality of the problem for $x = \frac{1}{3}$, $y = \frac{2}{3}$ and $z = 1$, as well as for $x = 0$, $y = \frac{1}{3}$, and $z = \frac{2}{3}$, we obtain

$$\frac{1}{2} \le \frac{1-b}{b-a} \le 2, \qquad \frac{1}{2} \le \frac{b-a}{a} \le 2$$

If $a < 0$, we would have $b - a < 0$ and $b < 0$. In addition, we would have $1 - b < 0$ or $b > 1$. A similar contradiction would be implied by the assumption $b - a < 0$. So $a > 0$ and $b - a > 0$, so

$$\frac{1}{3}\left(\frac{2}{3}a + \frac{1}{3}\right) \le a \le \frac{2}{3}\left(\frac{1}{3}a + \frac{2}{3}\right)$$

or $a \le 2b - 2a$, $b - a \le 2a$, $b - a \le 2 - 2b$, and $1 - b \le 2b - 2a$. Of these inequalities the first and third imply $3a \le 2b$ and $3b \le 2 + a$. Eliminate b to obtain $3a \le \frac{4}{3} + \frac{2a}{3}$, $a \le \frac{4}{7}$. In a corresponding manner, the second and fourth inequality imply $1 + 2a \le 3b$ and $b \le 3a$, from which $1 \le 7a$ or $\frac{1}{7} \le a$ follows. [The bounds can be improved. In fact the sharp lower and upper bounds for a are known to be $\frac{4}{27}$ and $\frac{76}{135}$.]

01.1. *Let A be a finite collection of squares in the coordinate plane such that the vertices of all squares that belong to A are (m, n), $(m+1, n)$, $(m, n+1)$, and $(m+1, n+1)$ for some integers m and n. Show that there exists a subcollection B of A such that B contains at least 25 % of the squares in A, but no two of the squares in B have a common vertex.*

Solution. Divide the plane into two sets by painting the strips of squares parallel to the y axis alternately red and green. Denote the sets of red and green squares by R and G, respectively. Of the sets $A \cap R$ and $A \cap G$ at least one contains at least one half of the squares in A. Denote this set by A_1. Next partition the strips of squares which contain squares of A_1 into two sets E and F so that each set contains every second square of A_1 on each strip. Now neither of the dets E and F has a common point with a square in the same set. On the other hand, at least one of the sets $E \cap A_1$, $F \cap A_1$ contains at least one half of the squares in A_1 and thus at least one quarter of the sets in A. This set is good for the required set B.

01.2. *Let f be a bounded real function defined for all real numbers and satisfying for all real numbers x the condition*

$$f\left(x+\frac{1}{3}\right)+f\left(x+\frac{1}{2}\right)=f(x)+f\left(x+\frac{5}{6}\right).$$

Show that f is periodic. (A function f is bounded, if there exists a number L such that $|f(x)| < L$ for all real numbers x. A function f is periodic, if there exists a positive number k such that $f(x+k) = f(x)$ for all real numbers x.)

Solution. Let $g(6x) = f(x)$. Then g is bounded, and

$$g(t+2)=f\left(\frac{t}{6}+\frac{1}{3}\right), \quad g(t+3)=f\left(\frac{t}{6}+\frac{1}{2}\right),$$

$$g(t+5)=f\left(\frac{t}{6}+\frac{5}{6}\right), \quad g(t+2)+g(t+3)=g(t)+g(t+5),$$

$$g(t+5)-g(t+3)=g(t+2)-g(t)$$

for all real numbers t. But then

$$\begin{aligned}
&g(t+12)-g(6)\\
&=g(t+12)-g(t+10)+g(t+10)-g(t+8)+g(t+8)-g(t+6)\\
&=g(t+9)-g(t+7)+g(t+7)-g(t+5)+g(t+5)-g(t+3)\\
&=g(t+6)-g(t+4)+g(t+4)-g(t+2)+g(t+2)-g(t)\\
&=g(t+6)-g(t).
\end{aligned}$$

By induction, then $g(t+6n)-g(t) = n(g(t+6)-g(0))$ for all positive integers n. Unless $g(t+6)-g(t) = 0$ for all real t, g cannot be bounded. So g has to be periodic with 6 as a period, whence f is periodic, with 1 as a period.

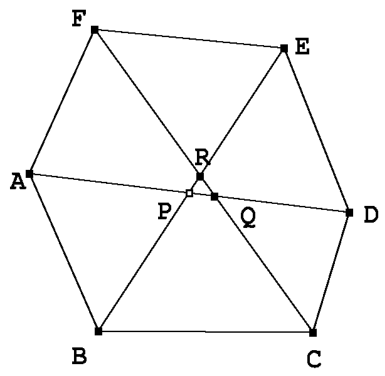


Figure 12.

01.3. *Determine the number of real roots of the equation*

$$x^8 - x^7 + 2x^6 - 2x^5 + 3x^4 - 3x^3 + 4x^2 - 4x + \frac{5}{2} = 0.$$

Solution. Write

$$\begin{aligned} &x^8 - x^7 + 2x^6 - 2x^5 + 3x^4 - 3x^3 + 4x^2 - 4x + \frac{5}{2} \\ &\quad = x(x-1)(x^6 + 2x^4 + 3x^2 + 4) + \frac{5}{2}. \end{aligned}$$

If $x(x-1) \geq 0$, i.e. $x \leq 0$ or $x \geq 1$, the equation has no roots. If $0 < x < 1$, then $0 > x(x-1) = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} \geq -\frac{1}{4}$ and $x^6 + 2x^4 + 3x + 4 < 1 + 2 + 3 + 4 = 10$. The value of the left-hand side of the equation now is larger than $-\frac{1}{4} \cdot 10 + \frac{5}{2} = 0$. The equation has no roots in the interval (0, 1) either.

01.4. *Let $ABCDEF$ be a convex hexagon, in which each of the diagonals AD, BE, and CF divides the hexagon in two quadrilaterals of equal area. Show that AD, BE, and CF are concurrent.*

Solution. (See Figure 12.) Denote the area of a figure by $|\cdot|$. Let AD and BE intersect at P, AD and CF at Q, and BE and CF at R. Assume that P, Q, and R are different. We may assume that P lies between B and R, and Q lies between C and R. Both $|ABP|$ and $|DEP|$ differ from $\frac{1}{2}|ABCDEF|$ by $|BCDP|$. Thus ABP and DEP have equal area. Since $\angle APB = \angle DPE$, we have $AP \cdot BP = DP \cdot EP = (DQ + QP)(ER + RP)$. Likewise $CQ \cdot DQ = (AP+PQ)(FR+RQ)$ and $ER \cdot FR = (CQ+QR)(BP+PR)$. When we multiply the three previous equalities, we obtain $AP \cdot BP \cdot CQ \cdot DQ \cdot ER \cdot FR = DQ \cdot ER \cdot AP \cdot FR \cdot CQ \cdot BP +$ positive terms containing PQ, QR, and PR. This is a contradiction. So P, Q and R must coincide.

02.1. *The trapezium $ABCD$, where AB and CD are parallel and $AD < CD$, is inscribed in the circle c. Let DP be a chord of the circle, parallel to AC. Assume that the tangent to c at D meets the line AB at E and that PB and DC meet at Q. Show that $EQ = AC$.*

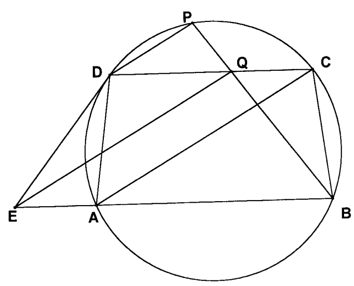


Figure 13.

Solution. (See Figure 13.) since $AD < CD$, $\angle PDC = \angle DCA < \angle DAC$. This implies that arc CP is smaller than arc CD, and P lies on that arc CD which does not include A and B. We show that the triangles ADE and CBQ are congruent. As a trapezium inscribed in a circle, $ABCD$ is isosceles (because $AB \| CD$, $\angle BAC = \angle DCA$, hence $BC = AD$). Because $DP \| AC$, $\angle PDC = \angle CAB$. But $\angle EDA = \angle CAB$ (angles subtending equal arcs) and $\angle PBC = \angle PDC$ (by the same argument). So $\angle EDA = \angle QBC$. Because $ABCD$ is an inscribed quadrilateral, $\angle EAD = 180° - \angle DAB = \angle DCB$. So $\angle EAD = \angle QCB$. The triangles ADE and CBQ are congruent (asa). But then $EA = QC$. As, in addition, $EA \| QC$, $EACQ$ is a parallelogram. And so $AC = EQ$, as opposite sides of a parallelogram.

02.2. *In two bowls there are in total N balls, numbered from 1 to N. One ball is moved from one of the bowls to the other. The average of the numbers in the bowls is increased in both of the bowls by the same amount, x. Determine the largest possible value of x.*

Solution. Consider the situation before the ball is moved from urn one to urn two. Let the number of balls in urn one be n, and let the sum of numbers in the balls in that urn be a. The number of balls in urn two is m and the sum of numbers b. If q is the number written in the ball which was moved, the conditions of the problem imply

$$\begin{cases} \dfrac{a-q}{n-1} = \dfrac{a}{n} + x, \\ \dfrac{b+q}{m+1} = \dfrac{b}{m} + x \end{cases}$$

or

$$\begin{cases} a = nq + n(n-1)x \\ b = mq - m(m+1)x. \end{cases}$$

Because $n + m = N$ and $a + b = \dfrac{1}{2}N(N+1)$, we obtain

$$\frac{1}{2}N(N+1) = Nq + x(n^2 - m^2 - N) = Nq + xN(n-m-1)$$

and $q = \frac{1}{2}(N+1) - x(n-m-1)$, $b = \dfrac{1}{2}m(N+1) - xmn$. But $b \ge 1+2+\cdots+m = \dfrac{1}{2}m(m+1)$. So $\dfrac{1}{2}(N+1) - xn = \dfrac{1}{2}(m+n+1) - xn \ge \dfrac{1}{2}(m+1)$ or $\dfrac{n}{2} - xn \ge 0$. Hence $x \le \dfrac{1}{2}$. The inequality is sharp or $x = \dfrac{1}{2}$, when the nubers in the balls in urn one are $m+1$, $m+2$, $\dots$, N, the numbers in urn two are 1, 2, $\dots$, m, and $q = m+1$.

02.3. *Let $a_1, a_2, \dots, a_n$, and $b_1, b_2, \dots, b_n$ be real numbers, and let $a_1, a_2, \dots, a_n$ be all different.. Show that if all the products*

$$(a_i + b_1)(a_i + b_2)\cdots(a_i + b_n),$$

$i = 1, 2, \dots, n$, are equal, then the products

$$(a_1 + b_j)(a_2 + b_j)\cdots(a_n + b_j),$$

$j = 1, 2, \dots, n$, are equal, too.

Solution. Let $P(x) = (x+b_1)(x+b_2)\cdots(x+b_n)$. Let $P(a_1) = P(a_2) = \dots = P(a_n) = d$. Thus $a_1, a_2, \dots, a_n$ are the roots of the n:th degree polynomial equation $P(x) - d = 0$. Then $P(x) - d = c(x-a_1)(x-a_2)\cdots(x-a_n)$. Clearly the n:th degree terms of $P(x)$ and $P(x) - d$ are equal. So $c = 1$. But $P(-b_j) = 0$ for each b_j. Thus for every j,

$$\begin{aligned} -d &= (-b_j - a_1)(-b_j - a_2)\cdots(-b_j - a_n) \\ &= (-1)^n(a_1 + b_j)(a_2 + b_j)\cdots(a_n + b_j), \end{aligned}$$

and the claim follows.

02.4. *Eva, Per and Anna play with their pocket calculators. They choose different integers and check, whether or not they are divisible by 11. They only look at nine-digit numbers consisting of all the digits 1, 2, $\dots$, 9. Anna claims* *that the probability of such a number to be a multiple of 11 is exactly* 1/11. *Eva has a different opinion: she thinks the probability is less than* 1/11. *Per thinks the probability is more than* 1/11. *Who is correct?*

Solution. We write the numbers in consideration, $n = a_0 + 10a_1 + 10^2a_2 + \cdots + 10^8a_8$, in the form

$$\begin{gathered} a_0 + (11-1)a_1 + (99+1)a_2 + (1001-1)a_3 \\ +(9999+1)a_4 + (100001-1)a_5 + (999999+1)a_6 \\ +(10000001-1)a_7 + (99999999+1)a_8 \\ = (a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - a_7 + a_8) + 11k \\ = (a_0 + a_1 + \cdots + a_8) - 2(a_1 + a_3 + a_5 + a_7) + 11k \\ = 44 + 1 + 11k - 2(a_1 + a_3 + a_5 + a_7). \end{gathered}$$

So n is divisible by 11 if and only if $2(a_1 + a_3 + a_5 + a_7) - 1$ is divisible by 11. Let $s = a_1 + a_3 + a_5 + a_7$. Then $1+2+3+4 = 10 \le s \le 6+7+8+9 = 30$ and $19 \le 2s-1 \le 59$. The only multiples of 11 in the desired interval are 33 and 55, so $s = 17$ or $s = 28$. If $s = 17$, the smallest number in the set $A = \{a_1, a_3, a_5, a_7\}$ is either 1 or 2 ($3+4+5+6 = 18$). Checking the cases, we see that there are 9 possible sets A: $\{2, 4, 5, 6\}$, $\{2, 3, 5, 7\}$, $\{2, 3, 4, 8\}$, $\{1, 4, 5, 7\}$, $\{1, 3, 6, 7\}$, $\{1, 3, 5, 8\}$, $\{1, 3, 4, 9\}$, $\{1, 2, 6, 8\}$, and $\{1, 2, 5, 9\}$. If $s = 28$, the largest number in A is 9 ($5+6+7+8 = 26$) and the second largest 8 ($5+6+7+9 = 27$). The only possible A's are $\{4, 7, 8, 9\}$ and $\{5, 6, 8, 9\}$. The number of different ways to choose the set A is $\binom{9}{4} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{2 \cdot 3 \cdot 4} = 126$. Of these, the number of choices leading to a number which is a multiple of 11 is $9 + 2 = 11$. This means that the probability of picking a number which is divisible by 11 is $\frac{11}{126} < \frac{11}{121} = \frac{1}{11}$. So Eva's opinion is correct.

03.1. *Stones are placed on the squares of a chessboard having 10 rows and 14 columns. There is an odd number of stones on each row and each column. The squares are coloured black and white in the usual fashion. Show that the number of stones on black squares is even. Note that there can be more than one stone on a square.*

Solution. Changing the order of rows or columns does not influence the number of stones on a row, on a column or on black squares. Thus we can order the rows and columns in such a way that the 5×7 rectangles in the upper left and lower right corner are black and the other two 5×7 rectangles are white. If the number of stones on black squares would be odd, then one of the black rectangles would have an odd number of stones while the number of stones on the other would be even. Since the number of stones is even, one of the white rectangles would have an odd number of stones and the other an even number. But this would imply either a set of five rows or a set of seven columns with an even number of stones. But this is not possible, because every row and column has an odd number of stones. So the number of stones on black squares has to be even.

03.2. *Find all triples of integers* $(x,\, y,\, z)$ *satisfying*

$$x^3 + y^3 + z^3 - 3xyz = 2003.$$

Solution. It is a well-known fact (which can be rediscovered e.g. by noticing that the left hand side is a polynomial in x having $-(y+z)$ as a zero) that

$$\begin{aligned} x^3+y^3+z^3-3xyz &= (x+y+z)(x^2+y^2+z^2-xy-yz-zx) \\ &= (x+y+z)\frac{(x-y)^2+(y-z)^2+(z-x)^2}{2}. \end{aligned}$$

The second factor in the right hand side is non-negative. It is not hard to see that 2003 is a prime. So the solutions of the equation either satisfy

$$\begin{cases} x+y+z=1 \\ (x-y)^2+(y-z)^2+(z-x)^2=4006 \end{cases}$$

or

$$\begin{cases} x+y+z=2003 \\ (x-y)^2+(y-z)^2+(z-x)^2=2 \end{cases}$$

Square numbers are $\equiv 0$ or $\equiv 1 \bmod 3$. So in the first case, exactly two of the squares $(x-y)^2$, $(y-z)^2$, and $(z-x)^2$ are multiples of 3. Clearly this is not possible. So we must have $x+y+z=2003$ and $(x-y)^2+(y-z)^2+(z-x)^2=2$. This is possible if and only if one of the squares is 0 and two are 1's. So two of x, y, z have to be equal and the third must differ by 1 of these. This means that two of the numbers have to be 668 and one 667. A substitution to the original equation shows that this necessary condition is also sufficient.

03.3. *The point D inside the equilateral triangle $\triangle ABC$ satisfies $\angle ADC = 150°$. Prove that a triangle with side lengths $|AD|$, $|BD|$, $|CD|$ is necessarily a right-angled triangle.*

Solution. (See Figure 14.) We rotate the figure counterclockwise $60°$ around C. Because ABC is an equilateral triangle, $\angle BAC = 60°$, so A is mapped on B. Assume D maps to E. The properties of rotation imply $AD = BE$ and $\angle BEC = 150°$. Because the triangle DEC is equilateral, $DE = DC$ and $\angle DEC = 60°$. But then $\angle DEB = 150° - 60° = 90°$. So segments having the lengths as specified in the problem indeed are sides of a right triangle.

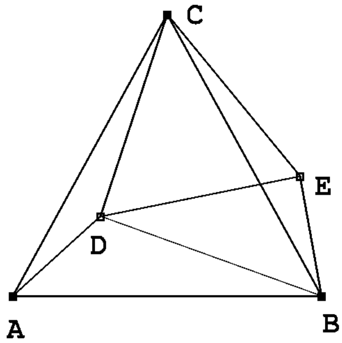


Figure 14.

03.4. *Let* $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ *be the set of non-zero real numbers. Find all functions* $f : \mathbb{R}^* \to \mathbb{R}^*$ *satisfying*

$$f(x) + f(y) = f(xy\,f(x+y)),$$

for $x,\ y \in \mathbb{R}^*$ *and* $x + y \neq 0$.

Solution. If $x \neq y$, then

$$f(y) + f(x-y) = f(y(x-y)f(x)).$$

Because $f(y) \neq 0$, we cannot have $f(x-y) = f(y(x-y)f(x))$ or $x - y = y(x-y)f(x)$. So for all $x \neq y$, $yf(x) \neq 1$. The only remaining possibility is $f(x) = \dfrac{1}{x}$. – One easily checks that f, $f(x) = \dfrac{1}{x}$, indeed satisfies the original functional equation.

04.1. *27 balls, labelled by numbers from 1 to 27, are in a red, blue or yellow bowl. Find the possible numbers of balls in the red bowl, if the averages of the labels in the red, blue, and yellow bowl are 15, 3 ja 18, respectively.*

Solution. Let R, B, and Y, respectively, be the numbers of balls in the red, blue, and yellow bowl. The mean value condition implies $B \le 5$ (there are at most two balls with a number < 3, so there can be at most two balls with a number > 3). R, B and Y satisfy the equations

$$\begin{aligned} R + B + Y &= 27 \\ 15R + 3S + 18Y &= \sum_{j=1}^{27} j = 14 \cdot 27 = 378. \end{aligned}$$

We eliminate S to obtain $4R + 5Y = 99$. By checking the possibilities we note that the pairs of positive integers satisfying the last equation are $(R, Y) = (21, 3)$, $(16, 7)$, $(11, 11)$, $(6, 15)$, and $(1, 19)$. The last two, however, do not satisfy $B = 27-(R+Y) \le 5$. We still have to ascertain that the three first alternatives are possible. In the case $R = 21$ we can choose the balls $5, 6, \ldots, 25$, in the red bowl, and 2, 3 and 4 in the blue bowl; if $P = 16$, $7, 8, \ldots, 14, 16, 17, \ldots, 23$, can go to the red bowl and 1, 2, 4 and 5 in the blue one, and if $P = 11$, the red bowl can have balls $10, 11, \ldots 20$, and the blue one 1, 2, 3, 4, 5. The red bowl can contain 21, 16 or 11 balls.

04.2. *Let $f_1 = 0$, $f_2 = 1$, and $f_{n+2} = f_{n+1} + f_n$, for $n = 1, 2, \ldots$, be the Fibonacci sequence. Show that there exists a strictly increasing infinite arithmetic sequence none of whose numbers belongs to the Fibonacci sequence. [A sequence is arithmetic, if the difference of any of its consecutive terms is a constant.]*

Solution. The Fibonacci sequence modulo any integer $n > 1$ is periodic. (Pairs of residues are a finite set, so some pair appears twice in the sequence, and the sequence from the second appearance of the pair onwards is a copy of the sequence from the first pair onwards.) There are integers for which the Fibonacci residue sequence does not contain all possible residues. For instance modulo 11 the sequence is 0, 1, 1, 2, 3, 5, 8, 2, 10, 1, 0, 1, 1, ... Wee see that the number 4 is missing. It follows that no integer of the form $4 + 11k$ appears in the Fibonacci sequence. But here we have an arithmetic sequence of the kind required.

04.3. *Let x_{11}, x_{21}, ..., x_{n1}, $n > 2$, be a sequence of integers. We assume that all of the numbers x_{i1} are not equal. Assuming that the numbers x_{1k}, x_{2k}, ..., x_{nk} have been defined, we set*

$$x_{i,k+1} = \frac{1}{2}(x_{ik} + x_{i+1,k}),\ i = 1, 2, \ldots, n-1,$$
$$x_{n,k+1} = \frac{1}{2}(x_{nk} + x_{1k}).$$

Show that for n odd, x_{jk} is not an integer for some j, k. Does the same conclusion hold for n even?

Solution. We compute the first index modulo n, i.e. $x_{1k} = x_{n+1,k}$. Let $M_k = \max_j x_{jk}$ and $m_k = \min_j x_{jk}$. Evidently (M_k) is a non-increasing and (m_k) a non-decreasing sequence, and $M_{k+1} = M_k$ is possible only if $x_{jk} = x_{j+1,k} = M_k$ for some j. If exactly p consequtive numbers x_{jk} equal M_k, then exactly $p-1$ consequtive numbers $x_{j,k+1}$ equal M_{k+1} which is equal to M_k. So after a finite number of steps we arrive at the situation $M_{k+1} < M_k$. In a corresponding manner we see that $m_{k+1} > m_k$ for some k's. If all the numbers in all the sequences are integers, then all m_k's and M_k's are integers. So after a finite number of steps $m_k = M_k$, and all numbers x_{jk} are equal. Then $x_{1,k-1} + x_{2,k-1} = x_{2,k-1} + x_{3,k-1} = \cdots = x_{n-1,k-1} + x_{n,k-1} = x_{n,k-1} + x_{1,k-1}$. If n is odd, then $x_{1,k-1} = x_{3,k-1} = \cdots = x_{n,k-1}$ and $x_{1,k-1} = x_{n-1,k-1} = \cdots = x_{2,k-1}$. But then we could show in a similar way that all numbers $x_{j,k-2}$ are equal and finally that all numbers $x_{j,1}$ are equal, contrary to the assumption. If n is even, then all $x_{i,k}$'s can be integers. Take, for instance, $x_{1,1} = x_{3,1} = \cdots = x_{n-1,1} = 0$, $x_{2,1} = x_{4,1} = \cdots = x_{n,1} = 2$. Then every $x_{j,k} = 1$, $k \geq 2$.

04.4. *Let a, b, and c be the side lengths of a triangle and let R be its circumradius. Show that*

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \geq \frac{1}{R^2}.$$

Solution 1. By the well-known (Euler) theorem, the inradius r and circumradius R of any triangle satisfy $2r \leq R$. (In fact, $R(R-2r) = d^2$, where d is the distance between the incenter and circumcenter.) The area S of a triangle can be written as

$$A = \frac{r}{2}(a+b+c),$$

and, by the sine theorem, as

$$A = \frac{1}{2}ab\sin\gamma = \frac{1}{4}\frac{abc}{R}.$$

Combining these, we obtain

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{a+b+c}{abc} = \frac{2A}{r} \cdot \frac{1}{4RA} = \frac{1}{2rR} \geq \frac{1}{R^2}.$$

Solution 2. Assume $a \leq b \leq c$. Then $b = a + x$ and $c = a + x + y$, $x \geq 0$, $y \geq 0$. Now $abc - (a+b-c)(a-b+c)(-a+b+c) = a(a+x)(a+x+y) - (a-y)(a+2x+y)(a+y) = ax^2 + axy + ay^2 + 2xy^2 + y^3 \geq 0$. So $abc(a+b+c) \geq (a+b+c)(a+b-c)(a-b+c)(-a+b+c) = 16A^2$, where the last inequality is implied by Heron's formula. When we substitute $A = \dfrac{abc}{4R}$ (see Solution 1) we obtain, after simplification,

$$a + b + c \geq \frac{abc}{R^2},$$

which is equivalent to the claim.

05.1. *Find all positive integers k such that the product of the digits of k, in the decimal system, equals*

$$\frac{25}{8}k - 211.$$

Solution. Let

$$a = \sum_{k=0}^{n} a_k 10^k, \quad 0 \le a_k \le 9, \text{ for } 0 \le k \le n-1,\ 1 \le a_n \le 9.$$

Set

$$f(a) = \prod_{k=0}^{n} a_k.$$

Since

$$f(a) = \frac{25}{8}a - 211 \ge 0,$$

$a \ge \frac{8}{25} \cdot 211 = \frac{1688}{25} > 66$. Also, $f(a)$ is an integer, and gcf(8, 25) = 1, so $8 \mid a$. On the other hand,

$$f(a) \le 9^{n-1}a_n \le 10^n a_n \le a.$$

So

$$\frac{25}{8}a - 211 \le a$$

or $a \le \frac{8}{17} \cdot 211 = \frac{1688}{17} < 100$. The only multiples of 8 between 66 and 100 are 72, 80, 88, and 96. Now $25 \cdot 9 - 211 = 17 = 7 \cdot 2$, $25 \cdot 10 - 211 = 39 \ne 8 \cdot 0$, $25 \cdot 11 - 211 = 64 = 8 \cdot 8$, and $25 \cdot 12 - 211 = 89 \ne 9 \cdot 6$. So 72 and 88 are the numbers asked for.

05.2. *Let a, b, and c be positive real numbers. Prove that*

$$\frac{2a^2}{b+c}+\frac{2b^2}{c+a}+\frac{2c^2}{a+b}\ge a+b+c.$$

Solution 1. Use brute force. Removing the denominators and brackets and combining simililar terms yields the equivalent inequality

$$\begin{gathered}0\le 2a^4+2b^4+2c^4+a^3b+a^3c+ab^3+b^3c+ac^3+bc^3\\ -2a^2b^2-2b^2c^2-2a^2c^2-2abc^2-2ab^2c-2a^2bc\\ =a^4+b^4-2a^2b^2+b^4+c^4-2b^2c^2+c^4+a^4-2a^2c^2\\ +ab(a^2+b^2-2c^2)+bc(b^2+c^2-2a^2)+ca(c^2+a^2-2b^2)\\ =(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2\\ +ab(a-b)^2+bc(b-c)^2+ca(c-a)^2\\ +ab(2ab-2c^2)+bc(2bc-2a^2)+ca(2ca-2b^2).\end{gathered}$$

The six first terms on the right hand side are non-negative and the last three can be written as

$$\begin{gathered}2a^2b^2-2abc^2+2b^2c^2-2a^2bc+2c^2a^2-2ab^2c\\ =a^2(b^2+c^2-2bc)+b^2(a^2+c^2-2ac)+c^2(a^2+b^2-2ab)\\ =a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2\ge 0.\end{gathered}$$

So the original inequality is true.

Solution 2. The inequality is equivalent to

$$\begin{gathered}2\left(a^2(a+b)(a+c)+b^2(b+c)(b+a)+c^2(c+a)(c+b)\right)\\ \ge (a+b+c)(a+b)(b+c)(c+a).\end{gathered}$$

The left hand side can be factored as $2(a+b+c)(a^3+b^3+c^3+abc)$. Because $a+b+c$ is positive, the inequality is equivalent to

$$2(a^3+b^3+c^3+abc)\ge (a+b)(b+c)(c+a).$$

After expanding the right hand side and subtracting $2abc$, we get the inequality

$$2(a^3+b^3+c^3) \geq (a^2b+b^2c+c^2a)+(a^2c+b^2a+c^2b),$$

still equivalent to the original one. But we now have two instances of the well-known inequality $x^3+y^3+z^3 \geq x^2y+y^2z+z^2x$ or $x^2(x-y)+y^2(y-z)+z^2(z-x) \geq 0$. [Proof: We may assume $x \geq y$, $x \geq z$. If $y \geq z$, write $z-x = z-y+y-z$ to obtain the equivalent and true inequality $(y^2-z^2)(y-z)+(x^2-z^2)(x-y) \geq 0$, if $z \geq y$, similarly write $x-y = x-z+z-y$, and get $(x^2-z^2)(x-z)+(x^2-y^2)(z-y) \geq 0$.]

Solution 3. The original inequality is symmetric in a, b, c. So we may assume $a \geq b \geq c$, which implies

$$\frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}.$$

The power mean inequality gives

$$\frac{a^2+b^2+c^2}{3} \geq \left(\frac{a+b+c}{3}\right)^2.$$

We combine this and the Chebyshev inequality to obtain

$$\begin{aligned} &\frac{2a^2}{b+c}+\frac{2b^2}{c+a}+\frac{2c^2}{a+b} \\ \geq\ &\frac{2}{3}(a^2+b^2+c^2)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right) \\ \geq\ &\frac{2}{9}(a+b+c)^2\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right). \end{aligned}$$

To complete the proof, we have to show that

$$2(a+b+c)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right) \geq 9.$$

But this is equivalent to the harmonic–arithmetic mean inequality

$$\frac{3}{\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}} \le \frac{x+y+z}{3},$$

with $x = a + b$, $y = b + c$, $z = c + a$.

05.3 *There are 2005 young people sitting around a (large!) round table. Of these at most 668 are boys. We say that a girl G is in a strong position, if, counting from G to either direction at any length, the number of girls is always strictly larger than the number of boys. (G herself is included in the count.) Prove that in any arrangement, there always is a girl in a strong position.*

Solution. Assume the number of girls to be g and the number of boys b. Call a position clockwise fairly strong, if, counting clockwise, the number of girls always exceeds the number of boys. No girl immediately followed by a boy has a fairly strong position. But no pair consisting of a girl and a boy following her has any effect on the fairly strongness of the other positions. So we may remove all such pairs. so we are left with at least $g - b$ girls, all in a clockwise fairly strong position. A similar count of counterclockwise fairly strong positions can be given, yielding at least $g - b$ girls in such a position. Now a sufficient condition for the existence of a girl in a strong position is that the sets consisting of the girls in clockwise and counterclockwise fairly strong position is nonempty. This is certainly true if $2(g-b) > g$, or $g > 2b$. With the numbers in the problem, this is true.

05.4. *The circle $\mathcal{C}_1$ is inside the circle $\mathcal{C}_2$, and the circles touch each other at A. A line through A intersects $\mathcal{C}_1$ also at B and $\mathcal{C}_2$ also at C. The tangent to $\mathcal{C}_1$ at B intersects $\mathcal{C}_2$ at D and E. The tangents of $\mathcal{C}_1$ passing through C touch $\mathcal{C}_1$ at F and G. Prove that D, E, F, and G are concyclic.*

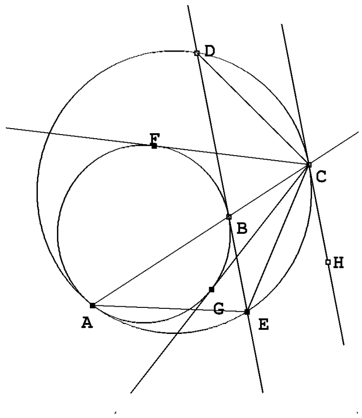


Figure 15.

Solution. (See Figure 15.) Draw the tangent CH to $\mathcal{C}_2$ at C. By the theorem of the angle between a tangent and chord, the angles ABH and ACH both equal the angle at A between BA and the common tangent of the circles at A. But this means that the angles ABH and ACH are equal, and $CH \| BE$. So C is the midpoint of the arc DE. This again implies the equality of the angles CEB and BAE, as well as $CE = CD$. So the triangles AEC, CEB, having also a common angle ECB, are similar. So

$$\frac{CB}{CE} = \frac{CE}{AC},$$

and $CB \cdot AC = CE^2 = CD^2$. But by the power of a point theorem, $CB \cdot CA = CG^2 = CF^2$. We have in fact proved $CD = CE = CF = CG$, so the four points are indeed concyclic.

06.1 *Let B and C be points on two fixed rays emanating from a point A such that $AB + AC$ is constant. Prove that there exists a point $D \neq A$ such that the circumcircles of*

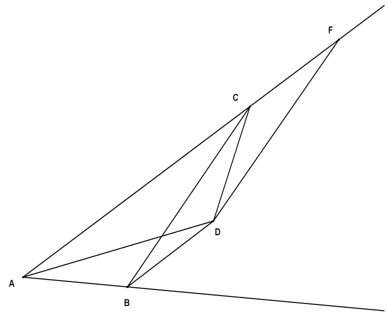


Figure 16.

the triangels ABC pass through D for every choice of B and C.

Solution. (See Figure 16.) Let E and F be the points on rays AB and AC, respectively, such that $AE = AF = AB + AC$. Let the perpendicular bisectors of the segments AE and AF intersect at D. It is easy to see, for instance from the right triangles with AD as the common hypothenuse and the projections of AD on AB and AC as legs, that D lies on the angle bisector of angle BAC. Moreover, $\angle ADF = 180° - 2 \cdot \angle CAD = 180° - \angle BAC$. The triangle ADF is isosceles, so $\angle BAD = \angle DAC = \angle CFD$ and $AD = DF$ in the triangles ABD and DCF. Moreover, we know that $CF = AF - AC = AB$. The triangles ADB and FDC are congruent (sas). So $\angle BDA = \angle CDF$. But this implies $\angle BDC = \angle ADF = 180° - \angle BAC$. This is sufficient for $ABDC$ to be an inscribed quadrilateral, and the claim has been proved.

06.2. *The real numbers x, y and z are not all equal and* *satisfy*

$$x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x} = k.$$

Determine all possible values of k.

Solution. Let $(x,\, y,\, z)$ be a solution of the system of equations Since

$$x = k - \frac{1}{y} = \frac{ky - 1}{y} \quad \text{and} \quad z = \frac{1}{k - y},$$

the equation

$$\frac{1}{k - y} + \frac{y}{ky - 1} = k,$$

to be simplified into

$$(1 - k^2)(y^2 - ky + 1) = 0,$$

is true. So either $|k| = 1$ or

$$k = y + \frac{1}{y}.$$

The latter alternative, substituted to the original equations, yields immediately $x = y$ and $z = y$. So $k = \pm 1$ is the only possibility. If $k = 1$, for instance $x = 2$, $y = -1$ and $z = \dfrac{1}{2}$ is a solution; if $k = -1$, a solution is obtained by reversing the signs for a solution with $k = 1$. So $k = 1$ and $k = -1$ are the only possible values for k.

06.3. *A sequence of positive integers $\{a_n\}$ is given by*

$$a_0 = m \qquad \text{and} \qquad a_{n+1} = a_n^5 + 487$$

for all $n \geq 0$. Determine all values of m for which the sequence contains as many square numbers as possible.

Solution. Consider the expression $x^5 + 487$ modulo 4. Clearly $x \equiv 0 \Rightarrow x^5 + 487 \equiv 3$, $x \equiv 1 \Rightarrow x^5 + 487 \equiv 0$; $x \equiv 2 \Rightarrow x^5 + 487 \equiv 3$, and $x \equiv 3 \Rightarrow x^5 + 487 \equiv 2$. Square numbers are always $\equiv 0$ or $\equiv 1 \mod 4$. If there is an even square in the sequence, then all subsequent numbers of the sequence are either $\equiv 2$ or $\equiv 3 \mod 4$, and hence not squares. If there is an odd square in the sequence, then the following number in the sequence can be an even square, but then none of the other numbers are squares. So the maximal number of squares in the sequence is two. In this case the first number of the sequence has to be the first square, since no number of the sequence following another one satisfies $x \equiv 1 \mod 4$. We have to find numbers k^2 such that $k^{10} + 487 = n^2$. We factorize $n^2 - k^{10}$. Because 487 is a prime, $n - k^5 = 1$ and $n + k^5 = 487$ or $n = 244$ and $k = 3$. The only solution of the problem thus is $m = 3^2 = 9$.

06.4. *The squares of a* 100×100 *chessboard are painted with* 100 *different colours. Each square has only one colour and every colour is used exactly 100 times. Show that there exists a row or a column on the chessboard in which at least 10 colours are used.*

Solution. Denote by R_i the number of colours used to colour the squares of the i'th row and let C_j be the number of colours used to colour the squares of the j'th column. Let r_k be the number of rows on which colour k appears and let c_k be the number of columns on which colour k appears. By the arithmetic-geometric inequality, $r_k + c_k \geq 2\sqrt{r_k c_k}$. Since colour k appears at most c_k times on each of the r_k columns on which it can be found, $c_k r_k$ must be at least the total number of occurences of colour k, which equals 100. So $r_k + c_k \geq 20$. In the sum $\sum_{i=1}^{100} R_i$, each colour k contributes r_k times and in the sum $\sum_{j=1}^{100} C_j$ each colour k contributes

c_k times. Hence

$$\sum_{i=1}^{100} R_i + \sum_{j=1}^{100} C_j = \sum_{k=1}^{100} r_k + \sum_{k=1}^{100} c_k = \sum_{k=1}^{100} (r_k + c_k) \geq 2000.$$

But if the sum of 200 positive integers is at least 2000, at least one of the summands is at least 10. The claim has been proved.

WINNERS OF THE NMC

The list gives, for each year, the highest scoring participant or participants in the competition.

1987: Elina Sihvola (Finland), Geir Agnarsson (Iceland), Richard Ehrenborg (Sweden)

1988: Patrik Andersson (Sweden), Daniel Bertilsson (Sweden), Mats Persson (Sweden)

1989: Mattias Jonsson (Sweden)

1990: Kimmo Uutela (Finland)

1991: Jan Kristian Haugland (Norway), Kong Xin-wei (Norway), Andreas Strömbergsson (Sweden)

1992: Jan Kristian Haugland (Norway)

1993: B. V. Halldórsson (Iceland)

1994: Bjarne Knudsen (Denmark)

1995: Uoti Urpala (Finland)

1996: Hans Rullgård (Sweden)

1997: Hans Rullgård (Sweden)

1998: Hannu Niemistö (Finland)

1999: David Kunszenti-Kovacs (Norway), David Rydh (Sweden), Hannu Niemistö (Finland), Jonas Sjöstrand (Sweden), Marteinn Thor Hardarson (Iceland)

2000: Øivind Grotmol (Norway), Jonas Sjöstrand (Sweden)

2001: Dávid Kunszenti-Kovács (Norway), Riikka Korte (Finland), Per-Anders Andersson (Sweden)

2002: Dávid Kunszenti-Kovács (Norway)

2003: Dávid Kunszenti-Kovács (Norway)

2004: David Ericsson (Sweden), Johan Bredberg (Sweden), Lauri Ahlroth (Finland), Miika Nikula (Finland) Paul Kjetel S. Lillemoen (Norway), Sebastian Dumitrescu (Finland)

2005: Sebastian Dumitrescu (Finland)

2006: Jørgen Vold Rennemo (Norway)