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1. The only numbers $n \leqslant 28$ satisfying the condition are $n \in \{2, 4, 10\}$. Let $n > 28$.

Note that the sequence a_i is symmetric with respect to $\frac{n}{2}$, so $a_i + a_{k+1-i} = n$.

If $2 \nmid n$, then $a_1 = 1$, $a_2 = 2$ and $3 \mid a_1 + a_2$. On the other hand, if $3 \mid n$, taking i such that $a_i < \frac{n}{2} < a_{i+1}$ would yield $3 \mid a_i + a_{i+1} = n$. Therefore $2 \mid n$ and $3 \nmid n$, implying that $a_2 = 3$. Then we also have $a_{k-1} = n - 3$ and $a_k = n - 1$.

If $n \equiv 2 \pmod 3$, then $3 \mid a_{k-1} + a_k = 2n - 4$. Therefore $n \equiv 1 \pmod 3$.

If $a_i + a_{i+1} \equiv 2 \pmod 3$ for some i, then $a_{k-i} + a_{k+1-i} = 2n - (a_i + a_{i+1}) \equiv 0 \pmod 3$. Hence $a_i + a_{i+1} \equiv 1 \pmod 3$ for all $i = 1, 2, \ldots, k-1$. Induction gives us

$$a_1 \equiv a_3 \equiv \cdots \equiv 1 \quad \text{and} \quad a_2 \equiv a_4 \equiv \cdots \equiv 0 \pmod 3. \qquad (*)$$

Since $(n, 9) = 1$, we have $a_4 = 9$, whereas $a_3 \equiv 1 \pmod 3$ implies that $a_3 = 7$. We deduce that $(n, 7) = 1$ and $(n, 5) \neq 1$, so $5 \mid n$.

Now we have $(n, 21) = (n, 27) = 1$, but the numbers 22, 24, 25 and 26 are not coprime to n, and by $(*)$, number 23 cannot occur among $a_1, \ldots, a_k$ either. It follows that 21 and 27 are consecutive terms in the sequence $a_1, \ldots, a_k$, but $3 \mid 21 + 27$, which is a contradiction.

Therefore, the only solutions are 2, 4 and 10.

2. **(a)** It suffices to show that every normalized sequence $a_1, a_2, \ldots, a_{2n+1}$ is embeddable in *some* interval of length $2 - \frac{1}{2^n}$. We use induction on n. Case $n = 0$ is trivial, so assume that $n \geqslant 1$. By the inductive hypothesis there is a sequence $x_0, x_1, \ldots, x_{2n-1} \in [0, 2 - \frac{1}{2^{n-1}}]$ such that $|x_i - x_{i-1}| = a_i$ for $i = 1, \ldots, 2n-1$. We assume w.l.o.g. that $x_{2n-1} \leqslant 1 - \frac{1}{2^n}$.

(1°) If $a_{2n} \geqslant \frac{1}{2^n}$, we can take $x_{2n} = x_{2n-1} + a_{2n} \in [1, 2 - \frac{1}{2^n}]$ and $x_{2n+1} = x_{2n} - a_{2n+1} \in [0, 2 - \frac{1}{2^n}]$, thus embedding the sequence in $[0, 2 - \frac{1}{2^n}]$.

(2°) If $a_{2n} < \frac{1}{2^n}$, we take $x_{2n} = x_{2n-1} - a_{2n} \in [-\frac{1}{2^n}, 1 - \frac{1}{2^n}]$ and $x_{2n+1} = x_{2n} + a_{2n+1}$, thus embedding the sequence in one of the intervals $[0, 2 - \frac{1}{2^n}]$ and $[-\frac{1}{2^n}, 2 - \frac{1}{2^{n-1}}]$.

(b) Denote $N = 3 \cdot 2^{n-1} - 1$. We claim that the following sequence of length $4n - 1$:

$$1,\ 1-\frac{1}{N},\ 1,\ 1-\frac{2}{N},\ 1,\ 1-\frac{2^2}{N},\ \ldots,\ 1,\ 1-\frac{2^{n-1}}{N},\ 1,\ 1-\frac{2^{n-2}}{N},\ 1,\ \ldots,\ 1-\frac{2}{N},\ 1,\ 1-\frac{1}{N},\ 1$$

cannot be embedded in the interval $(-1 + \frac{1}{2N}, 1 - \frac{1}{2N})$, which implies the statement. Assume the opposite. It follows by simple induction that:

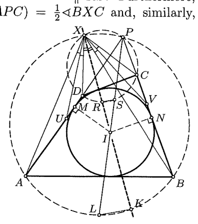
(i) $|x_{2i}| < 1 - \frac{2^{i+1}-1}{2N}$ and $|x_{2i+1}| > \frac{2^{i+1}-1}{2N}$ for $i = 0, \ldots, n-1$;

(ii) $|x_{2i}| < \frac{2^{2n-i}-1}{2N}$ and $|x_{2i+1}| > 1 - \frac{2^{2n-i}-1}{2N}$ for $i = n, \ldots, 2n-1$.

Thus for x_{4n-1} we arrive at a contradiction.

3. It follows from $\sphericalangle XAD = \sphericalangle XBC$ and $\sphericalangle XDP = \sphericalangle XCP$ that $\triangle XAD \sim \triangle XBC$. Let the bisector s_X of angle AXC meet the circles PAB and PCD again at K and R, and let the bisector s_P of angle APC meet the circles PAB and PCD again at L and S, respectively. Line s_P passes through the center I of circle k. We recall that $LA = LB = LI$ and $SC = SD = SI$.

Since $\sphericalangle ILK = \sphericalangle PXK = \sphericalangle PXR = \sphericalangle ISR$, we have $KL \parallel RS$. Furthermore, $\sphericalangle RXS = \sphericalangle RXC - \sphericalangle SPC = \frac{1}{2}(\sphericalangle AXC - \sphericalangle APC) = \frac{1}{2}\sphericalangle BXC$ and, similarly, $\sphericalangle LXK = \frac{1}{2}\sphericalangle BXC$. It follows that arcs KL and RS subtend equal angles in the circles PAB and PCD, as well as arcs LB and SD, so we have $\frac{KL}{RS} = \frac{LB}{SD} = \frac{LI}{SI}$. Hence $\triangle IKL \sim \triangle IRS$, which means that I lies on the line KR which bisects angle AXC. This completes the proof.



Second solution. Denote by U and V the intersections of the bisectors of angles AXD and BXC with AD and BC, respectively. As in the first solution, $\triangle XAD \sim \triangle XBC$ and hence $\sphericalangle XUP = \sphericalangle XVP$, so the points X, P, U and V lie on a single circle γ. We shall prove that I also lies on γ and that $IU = IV$. This will imply that I lies on the bisector of $\sphericalangle UXV$, which coincides with the bisector of $\sphericalangle AXC$.

Let M and N be the tangency points of k with sides AD and BC, respectively. Denote $AM = a$, $BN = b$, $CN = c$ and $DM = d$. Then $AB = a+b$, $CD = c+d$ and $AU : UD = (a+b) : (c+d)$, from which we find $AU = \frac{a+b}{a+b+c+d} \cdot AD = \frac{(a+b)(a+d)}{a+b+c+d}$; Analogously, $BV = \frac{(b+a)(b+c)}{a+b+c+d}$. It follows that $AM - AU = BV - BN = \frac{ac-bd}{a+b+c+d}$, so $MU = NV$ and hence $\triangle IMU$ i $\triangle INV$ are congruent and equally oriented. We infer that $IU = IV$ and $\sphericalangle UIV = \sphericalangle MIN = 180^\circ - \sphericalangle VPU$, i.e. I is the midpoint of arc UV of the circle $PXUV$.

Third solution. The following statement is known from projective geometry.

- *Desargue's Involution Theorem.* Let a quadrilateral $ABCD$ be inscribed in a conic γ. A line ℓ intersects AB, CD, BC, DA, AC, BD respectively at points X_1, X_2, Y_1, Y_2, Z_1, Z_2, and meets γ at W_1 and W_2. Then there is an involution on ℓ that maps $X_1 \leftrightarrow X_2$, $Y_1 \leftrightarrow Y_2$, $Z_1 \leftrightarrow Z_2$ and $W_1 \leftrightarrow W_2$.

Its dual statement, obtained by the polar map with respect to γ, is as follows:

- Let a conic γ be inscribed in a quadrilateral $ABCD$ with $AD \cap BC = \{P\}$ and $AB \cap CD = \{Q\}$. Let XU and XV be tangents from an arbitrary point X to γ. Then there is an involution on the pencil of lines through X that maps $XA \leftrightarrow XC$, $XB \leftrightarrow XD$, $XP \leftrightarrow XQ$ and $XU \leftrightarrow XV$.

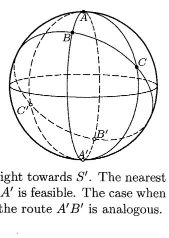
In our case, the angles AXC, BXD and PXQ have the common bisector s, so this involution must be the reflection in s. Therefore, the two tangents from X to k are symmetric with respect to s.

5. The answer is $n \leqslant 3$.

The gas station paired with station X will be denoted by X'.

For $n \leqslant 1$ the statement trivially holds. For $n = 2$, let $AB = A'B'$ be the smallest of all distances between two stations. Starting from A, the car can reach A', e.g. by the route $AB'A'$ (the case of route ABA' is similar), but in B there is enough gas to reach the nearest station, which is A, so the route $BAB'A'$ is feasible.

We will now show the statement for $n = 3$ and six stations A, A', B, B', C, C'. Let $AB = A'B'$ be the smallest of the distances between two stations and assume w.l.o.g. B is the nearest station to C. We partition the stations into sets $S = \{A, B, C\}$ and $S' = \{A', B', C'\}$. Starting from any station the car can reach the other set.



(1°) Suppose that the car starting from A cannot reach set S' by the route AB. Reaching S' is then impossible by the route AC as well - otherwise route $ABCS'$ would be feasible, because $BC \leqslant AC$ and the gas in B suffices to compensate for the usage on the route AB. Therefore, starting from A, the car must go straight towards S'. The nearest station in S' is C', so the entire route $CBAC'B'A'$ is feasible. The case when the car starting from A' cannot reach set S by the route $A'B'$ is analogous.

(2°) If case (1°) does not apply, let the car head from A straight to B. Since set S' is in the reach and $BC \leqslant d(B, S') = BC'$, the car can proceed from B to C. The gas in C will compensate for the usage on the route BC, and since $d(C, S') = CA' < d(B, S')$, set S' is still in the reach and the nearest station is A'. As before, the car can now go to B' and then to C', which results in the route $ABCA'B'C'$.

It remains to construct a counterexample for $n \geqslant 4$. Assuming that the semiperimeter of the planet is 1, arrange the stations $A_2, A_3, \ldots, A_n$ along the equator such that $A_2A_3 = A_3A_4 = \cdots = A_{n-1}A_n = d < \frac{1}{n-1}$ and place the station A_1 such that $A_1A_3 = d$ and $A_1A_2 = A_1A_4$. We again denote $S = \{A_1, \ldots, A_n\}$ and $S' = \{A'_1, \ldots, A'_n\}$. Supply the stations $A_1, A'_1, \ldots, A_{n-1}, A'_{n-1}$ with gas needed to drive the distance d, and the stations A_n and A'_n with gas needed to drive the distance $1 - (n-1)d$. From each station it is possible to reach the paired station: Indeed, the routes $A_1A_3A_4 \ldots A_nA'_2A'_3A'_1$ and $A_iA_{i+1} \ldots A_nA'_2A'_3 \ldots A'_i$ for $2 \leqslant i \leqslant n$ are all feasible. On the other hand, each of the stations $A_1, \ldots, A_{n-1}$ has only enough gas for reaching the nearest station, and in A_n and A'_n only enough for reaching the other set. So, in order to visit all stations, the car must pass w.l.o.g. the entire set S without using the gas in A_n, i.e. with the gas that is only enough to drive the distance $(n-1)d$, which is impossible.